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Solutions of the Dirac equation with an improved expression of the Rosen–Morse potential energy model including Coulomb-like tensor interaction

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Abstract: In this study, we obtain the approximate analytical solutions of the Dirac equation for an improved expression of the Rosen–Morse potential energy model including the Coulomb-like tensor under the condition of spin and pseudospin symmetry. The analytical approach of parametric generalization of the Nikiforov–Uvarov method has been applied to the problem and the problem is discussed in a quite detailed manner.

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Résumé : Nous obtenons ici des solutions analytiques approximatives de l'équation de Dirac pour une expression améliorée de l'énergie potentielle de Rosen–Morse, incluant le tenseur de type Coulomb, sous la condition de symétrie de spin et de pseudospin. Nous appliquons au problème l'approche analytique de la généralisation paramétrique de Nikiforov–Uvarov et nous proposons une analyse très détaillée du problème. [Traduit par la Rédaction]

1. Introduction

The discussion of the so-called spin and pseudospin symmetries of the Dirac equation is motivated by the notable phenomenological consequences of these symmetries in hadronic and nuclear spectroscopy. About three decades ago, the pseudospin symmetry was originally observed in spherical atomic nuclei [1, 2] and later used to explain features of deformed nuclei [3], including superdeformation [4], and to establish an effective shell model coupling scheme [5]. Based on the relativistic mean-field theory, Ginocchio [6] showed that the pseudospin symmetry in nuclei could arise from nucleons moving in a relativistic mean field that is nearly equal in magnitude to an attractive scalar potential $S(r)$ and repulsive vector potential $V(r)$. The sum of the vector and scalar potentials, $\Sigma(r) = V(r) + S(r)$, is small enough to consider the pseudospin symmetry slightly broken in nuclei. Meng et al. [7] showed that the pseudospin symmetry is exact under a more general condition, $d[V(r) + S(r)]/dr = 0$, and the quality of the pseudospin approximation in real nuclei is connected with the competition between the pseudocentrifugal barrier and the pseudospin–orbital potential. In the limit of small pseudospin–orbital potential, the pseudospin symmetry becomes exact. Generally speaking, the pseudospin symmetry occurs for $\Sigma(r) = V(r) + S(r) = \text{constant}$ in the Dirac equation [8].

In recent years, there has been considerable interest in studying the Dirac equation. The relativistic symmetries of the Dirac equation have been investigated under the Tietz potential for the scalar and vector interactions beside a Coulomb tensor term [9]. The analytical bound state solutions of the Dirac equation for the Manning–Rosen potential plus a ring-shaped potential in spin and pseudospin symmetries have been investigated very recently by a proper approximation to the centrifugal term [10]. Recently, Guo explored p-spin symmetry using the similarity renormalization group and showed explicitly the relativistic origin of the symmetry [11]. In recent years, the contribution of the isoscalar tensor coupling to the realization of pseudospin symmetry has been discussed by Alberto et al. [12].

Chen and Guo [13] investigated the evolution of the spin and pseudospin symmetries from the relativistic to the nonrelativistic and explored the relativistic relevance of the symmetries. By examining the zeros of Jost functions corresponding to the small components of Dirac wave functions and phase shifts of continuum states, Lu et al. [14] show that the pseudospin symmetry in single particle resonant states in nuclei is conserved when the attractive scalar and repulsive vector potentials have the same magnitude but opposite sign. The unbound solutions of generalized asymmetrical Hartmann potentials under the condition of the p-spin symmetry have been presented by mapping the wave functions of bound states in the complex momentum plane via the continuation method [15].

Despite all the attempts made by several authors to investigate these symmetries under a wide range of potentials (see refs. 9, 10, 16–34, and references therein), until now, there exists no work on an improved expression of the Rosen–Morse potential energy model. Very recently, Jia et al. [35] employed the dissociation energy and the equilibrium bound length for diatomic molecules as explicit parameters to generate an improved expression for the well-known Rosen–Morse potential. This potential can be written as

$$U_{\text{RM}}(r) = D_e \left(1 - \frac{b}{e^{2r/d} + 1} \right)^2 \quad b = e^{2r_e/d} + 1 \quad (1)$$

where D_e is the dissociation energy, r_e is the equilibrium bound length, and α denotes the range of the potential. This potential was constructed from the Rosen–Morse potential [36] by Jia et al. [35] on the basis that empirical potential energy function $U(r)$ should satisfy the following conditions [35]:

$$\begin{aligned} \frac{dU(r)}{dr} \Big|_{r=r_e} &= 0 & U(\infty) - U(r_e) &= D_e \quad \text{and} \\ \frac{d^2U(r)}{dr^2} \Big|_{r=r_e} &= k_e = 4\pi^2 \mu c^2 \omega_e^2 \end{aligned} \quad (2)$$

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where μ is the reduced mass of the diatomic molecule, c is the speed of light, and ω_e denotes the equilibrium harmonic vibrational frequency. This potential can be used to investigate polyatomic vibrational states, such as the vibrational states of NH_3 molecules, and to describe diatomic interaction [35–37].

It is therefore the main objective of the present work to investigate this potential under the mentioned symmetry limits. The organization of this paper is as follows. In Sect. 2 we give a brief overview of the Dirac bound state solutions including the Coulomb-like tensor. Section 3 is devoted to the approximate relativistic bound state solutions in the presence of this potential. Some numerical results and discussion are given in Sect. 4. Finally, we give a brief conclusion in Sect. 5.

2. Dirac bound state solutions including Coulomb-like tensor

In spherical coordinates, the Dirac equation for fermionic massive spin-1/2 particles interacting with arbitrary scalar potential $S(r)$, the time-component $V(r)$ of a four-vector potential and the tensor potential $U(r)$ can be expressed as [9, 22, 23, 38]

$$[\alpha \cdot \mathbf{p} + \beta(M + S(r)) - i\beta\alpha \cdot \hat{\mathbf{r}}U(r)]\Psi_{n\kappa}(\mathbf{r}) = [E - V(r)]\Psi_{n\kappa}(\mathbf{r}) \quad (3)$$

where E and M denote the relativistic energy of the system and the rest mass of the spin-half particle, respectively; $\mathbf{p} = -i\nabla$ is the three dimensional momentum operator; and α and β are 4×4 Dirac matrices, given as

$$\begin{aligned} \bar{\alpha} &= \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} & \beta &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} & \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \quad (4)$$

where I is the 2×2 unitary matrix and $\boldsymbol{\sigma}$ are the three-vector Pauli spin matrices. The eigenvalues of the spin-orbit coupling operator are $\kappa = [j + (1/2)] > 0$ and $\kappa = -[j + (1/2)] < 0$ for unaligned spin $j = \ell - (1/2)$ and the aligned spin $j = \ell + (1/2)$, respectively. The set (H^2, K, J^2, J_z) can be taken as the complete set of conservative quantities with $\mathbf{J}$ being the total angular momentum operator and $K = (\boldsymbol{\sigma} \cdot \mathbf{L} + 1)$ is the spin-orbit where $\mathbf{L}$ is the orbital angular momentum of the spherical nucleons that commutes with the Dirac Hamiltonian. Thus, the spinor wave functions can be classified according to their angular momentum, j ; the spin-orbit quantum number, κ ; and the radial quantum number, n . Hence, they can be written as

$$\Psi_{n\kappa}(\mathbf{r}) = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(\mathbf{r})Y_{jm}^\ell(\theta, \vartheta) \\ iG_{n\kappa}(\mathbf{r})Y_{jm}^{\bar{\ell}}(\theta, \vartheta) \end{pmatrix} \quad (5)$$

where $F_{n\kappa}(\mathbf{r})$ and $G_{n\kappa}(\mathbf{r})$ are the radial wavefunctions of the upper and lower spinor components, respectively. $Y_{jm}^\ell(\theta, \vartheta)$ and $Y_{jm}^{\bar{\ell}}(\theta, \vartheta)$ are the spherical harmonic functions coupled to the total angular momentum, j , and its projection, m , on the z -axis. The orbital and pseudo-orbital angular momentum quantum numbers for spin symmetry ℓ and pseudospin symmetry $\bar{\ell}$ refer to the upper and lower components, respectively. Substituting (5) into (3) yields the following coupled differential equations:

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r))G_{n\kappa}(r) \quad (6)$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r))F_{n\kappa}(r) \quad (7)$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$ are the difference and the sum potentials, respectively. By eliminating $G_{n\kappa}(r)$ in (6) and $F_{n\kappa}(r)$ in (7), we obtain two second-order nonlinear differential equations for the upper and lower radial spinor components as, respectively,

$$\begin{aligned} \left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - U^2(r) - \frac{dU(r)}{dr}\right. \\ \left. + \frac{d\Delta(r)/dr}{M + E_{n\kappa} - \Delta(r)}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)\right]F_{n\kappa}(r) \\ = [(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))]F_{n\kappa}(r) \end{aligned} \quad (8)$$

$$\begin{aligned} \left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) - U^2(r) + \frac{dU(r)}{dr}\right. \\ \left. + \frac{d\Sigma(r)/dr}{M - E_{n\kappa} + \Sigma(r)}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)\right]G_{n\kappa}(r) \\ = [(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))]G_{n\kappa}(r) \end{aligned} \quad (9)$$

where $\kappa(\kappa-1) = \bar{\ell}(\bar{\ell}+1)$ and $\kappa(\kappa+1) = \ell(\ell+1)$.

2.1. Spin symmetry limit

In the spin symmetry limit $d\Delta(r)/dr = 0$ or $\Delta(r) = C_s = \text{constant}$ (see refs. 9, 10, 16–26, and the references therein). By following the literature (see refs. 9, 10, 16–26, and the references therein), we take $\Sigma(r)$ as the improved expression of the Rosen–Morse potential energy model [35], that is,

$$\Sigma(r) = D_e \left(1 - \frac{b}{e^{2r/d} + 1}\right)^2 \quad (10)$$

and the Coulomb tensor potential can be written as

$$U(r) = -\frac{T}{r} \quad r \geq R_c \quad (11)$$

with

$$T = \frac{Z_a Z_b e^2}{4\pi\epsilon_0} \quad (12)$$

where R_c is the Coulomb radius; and Z_a and Z_b denote the charges of the projectile a and the target nuclei b , respectively [6, 8]. By imposing this symmetry on (8), we can have the following transformed equation:

$$\begin{aligned} \left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2} + D_e(M + E_{n\kappa} - C_s)\left(1 - \frac{b}{e^{2r/d} + 1}\right)^2\right. \\ \left. + \frac{2\kappa T}{r^2} + \frac{T}{r^2} + \frac{T^2}{r^2}\right]F_{n\kappa}(r) = [E_{n\kappa}^2 - M^2 + C_s(M - E_{n\kappa})]F_{n\kappa} \end{aligned} \quad (13)$$

where $\kappa = \ell$ and $\kappa = -\ell - 1$ for $\kappa < 0$ and $\kappa > 0$, respectively.

2.2. Pseudospin symmetry limit

The pseudospin symmetry when $d[V(r) + S(r)]/dr = d\Sigma(r)/dr = 0$ or, equivalently, $\Sigma(r) = C_{ps} = \text{const}$. Here our desire interaction is to take $\Delta(r)$ as the improved expression of the Rosen–Morse potential energy model and the tensor as the Coulomb-like potential, that is,

Table 1. The specific values of the parametric constants for the spin and pseudospin symmetric Dirac–Rosen–Morse problem.

Constant	Analytic value
a_4	0
a_5	1/2
a_6	(1/4) + ζ_1
a_7	$-\zeta_2$
a_8	ζ_3
a_9	$\zeta_1 + \zeta_2 + \zeta_3 + (1/4)$
a_{10}	$1 + 2\sqrt{\zeta_3}$
a_{11}	$-2 + 2(\sqrt{\zeta_1 + \zeta_2 + \zeta_3} - \sqrt{\zeta_3})$
a_{12}	$\sqrt{\zeta_3}$
a_{13}	$(1/2) + \sqrt{\zeta_3} + \sqrt{\zeta_1 + \zeta_2 + \zeta_3 + (1/4)}$

$$\Delta(r) = D_e \left(1 - \frac{b}{e^{2r/d} + 1} \right)^2 \quad \text{and} \quad U(r) = -\frac{T}{r} \quad r \geq R_c \quad (14)$$

By substituting (14) into (9), we can easily find

$$\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa - 1)}{r^2} - D_e(M - E_{n\kappa} + C_{ps}) \left(1 - \frac{b}{e^{2r/d} + 1} \right)^2 + \frac{2\kappa T}{r^2} - \frac{T}{r^2} + \frac{T^2}{r^2} \right] G_{n\kappa}(r) = [E_{n\kappa}^2 - M^2 - C_{ps}(M + E_{n\kappa})] G_{n\kappa} \quad (15)$$

where $\kappa = -\ell$ and $\kappa = \tilde{\ell} + 1$ for $\kappa < 0$ and $\kappa > 0$, respectively.

3. Approximate relativistic bound-states solutions

In this section, within the framework of the parametric generalization of the Nikiforov–Uvarov method, we shall approximately solve the Dirac equation with the improved expression of

$$\frac{4}{d^2} \frac{d^2 F_{n\kappa}(s)}{ds^2} + \frac{4}{sd^2} \frac{dF_{n\kappa}(s)}{ds} + \frac{1}{s^2(1+s)^2} \{ -(\delta D_0 + \gamma + \beta^2 - \delta D_1 - 2\gamma b + \delta D_2 + \gamma b^2)s^2 + s[\delta(D_1 - 2D_0) + 2\gamma(b - 1) - 2\beta^2] - (\delta D_0 + \gamma + \beta^2) \} F_{n\kappa}(s) = 0 \quad (18)$$

where the following notation has been introduced to avoid mathematical complexity:

$$\gamma = (M + E_{n\kappa} - C_s)D_e \quad \beta = \sqrt{(M - E_{n\kappa} - C_s)(M - E_{n\kappa})} \quad (19)$$

and $\delta = (\kappa + T + 1)(\kappa + T)$

Now, the solutions of (18) can be easily found by comparing it with (A2). Thus, the coefficients of a_i ($i = 1, 2, 3$) are as follows:

$$a_1 = 1 \quad a_2 = a_3 = -1 \quad (20)$$

It is also easy to obtain other parametric constants a_i ($i = 4, 5, 6, \dots, 13$) by using (A1) from Appendix A. A summary of the specific values of these coefficients a_i ($i = 4, 5, 6, \dots, 13$) is provided in Table 1. Having determined these constants, we can obtain the relativistic energy equation for this problem as

the Rosen–Morse potential energy model in the presence of the tensor potential.

3.1. Solutions for spin symmetry limit

To obtain the solution of (13), it is necessary to include some approximation to make the problem solvable. Lu [39] has introduced an approximation to the centrifugal term near the minimum point $r = r_e$ as

$$\frac{1}{r^2} \approx D_0 - D_1 \frac{\exp(-2r/d)}{1 + \exp(-2r/d)} + D_2 \left[\frac{\exp(-2r/d)}{1 + \exp(-2r/d)} \right]^2 \quad (16)$$

where

$$D_0 = \frac{1}{r_e} \left\{ 1 - \left[\frac{1 + \exp(-2r_e/d)}{-2r_e/d} \right]^2 \left[\frac{8r_e/d}{1 + \exp(-2r_e/d)} - 3 - \frac{2r_e}{d} \right] \right\}$$

$$D_1 = -\frac{2}{r_e} \left[\exp\left(\frac{2r_e}{d}\right) + 1 \right] \times \left\{ \frac{3[1 + \exp(-2r_e/d)]}{2r_e/d} - \left(3 + \frac{2r_e}{d} \right) \frac{[1 + \exp(-2r_e/d)]}{2r_e/d} \right\} \quad (17)$$

$$D_2 = \frac{[\exp(2r_e/d) + 1]^2}{r_e} \left[\frac{1 + \exp(-2r_e/d)}{2r_e/d} \right]^2 \times \left[3 + \frac{2r_e}{d} - \frac{4r_e/d}{1 + \exp(-2r_e/d)} \right]$$

and higher order terms are neglected. Now, with the help of (16) and by introducing a new parameter change, $s(r) = e^{-2r/d} \in [0, 1]$, which maintains the finiteness of the transformed wave functions on the boundary conditions, this allows us to decompose the spin-symmetric Dirac equation, (13), into the Schrödinger-like equation in the spherical coordinates for the upper-spinor component, $F_{n\kappa}(r)$,

$$-n^2 - n - \frac{1}{2} + (2n + 1) \left(\sqrt{\zeta_1 + \zeta_2 + \zeta_3} - \sqrt{\zeta_3} \right) - \zeta_2 - 2\zeta_3 + 2 \sqrt{\zeta_3 \left(\zeta_2 + \zeta_1 + \zeta_3 + \frac{1}{4} \right)} = 0 \quad (21)$$

or more explicitly as

$$\delta D_0 + \gamma + \beta^2 - \delta D_1 - 2\gamma b + \delta D_2 + \gamma b^2 - \frac{4}{d^2} \left(n + \frac{1}{2} - \frac{1}{2} \sqrt{\delta D_2 d^2 + \gamma b^2 d^2 + 1} + \frac{d}{2} \sqrt{\delta D_0 + \gamma + \beta^2} \right)^2 = 0 \quad (22)$$

Let us now obtain the un-normalized upper component of the spinor wave functions by using (A10) from Appendix A. Thus, the wave functions can be expressed in terms of the Jacobi polynomials as

Table 2. The relativistic bound state spin symmetric energy eigenvalues in units of fm⁻¹ of the improved expression of the Rosen–Morse potential energy model for several values of n and κ .

ℓ	$n, \kappa < 0$	$nL_{j=\ell+1/2}$	$C_s = 0$		$C_s = 5$	
			$E_{n,\kappa<0}(T=0)$	$E_{n,\kappa<0}(T=0.5)$	$E_{n,\kappa<0}(T=0)$	$E_{n,\kappa<0}(T=0.5)$
1	0, -2	0p _{3/2}	15.435 865 236 0	9.920 053 328 81	17.766 387 819 7	12.306 882 629 9
1	0, -2	0p _{3/2}	—	20.482 482 504 0	—	—
2	0, -3	0d _{5/2}	25.306 694 271 9	-32.957 292 093 8	27.604 868 224 2	22.789 301 216 8
2	0, -3	0d _{5/2}	-46.546 918 839 0	30.000 583 808 2	-42.724 013 026 7	-29.445 638 951 1
3	0, -4	0f _{5/2}	34.610 196 198 4	-62.677 905 209 6	36.910 768 553 0	32.297 896 695 9
3	0, -4	0f _{5/2}	-81.500 414 590 9	39.161 937 688 6	-77.235 371 822 4	-58.604 631 382 7
4	0, -5	0g _{9/2}	43.672 221 634 1	-103.092 083 293	45.984 639 418 4	41.467 912 838 3
4	0, -5	0g _{9/2}	-127.493 140 894	-0.045 871 275 54	-122.974 788 986	-98.682 629 756 9
1	1, -2	1p _{3/2}	15.694 507 311 9	10.242 884 330 3	18.008 614 664 7	12.599 288 176 9
1	1, -2	1p _{3/2}	—	—	-42.721 748 779 4	—
2	1, -3	1d _{5/2}	25.503 813 697 5	20.704 785 663 5	27.793 334 332 1	23.000 238 700 3
2	1, -3	1d _{5/2}	-46.544 824 826 5	-32.954 276 844 6	-77.233 682 766 6	-29.442 334 896 2
3	1, -4	1f _{7/2}	34.773 475 875 4	30.178 760 491 1	37.068 373 198 2	32.469 165 277 9
3	1, -4	1f _{7/2}	-81.498 710 376 5	-62.676 167 031 5	-77.233 682 766 6	-58.602 843 400 4
4	1, -5	1g _{9/2}	43.813 470 484 3	39.313 157 731 8	46.121 842 599 9	41.614 377 354 2
4	1, -5	1g _{9/2}	-127.491 139 500	-103.090 259 406	-122.972 831 579	-98.680 845 493 7
ℓ	$n, \kappa > 0$	$nL_{j=\ell-1/2}$	$E_{n,\kappa>0}(T=0)$	$E_{n,\kappa>0}(T=0.5)$	$E_{n,\kappa>0}(T=0)$	$E_{n,\kappa>0}(T=0.5)$
1	0, 1	0p _{1/2}	15.435 865 236 0	20.482 482 504 0	17.766 387 819 7	22.789 301 216 8
1	0, 1	0p _{1/2}	—	-32.957 292 093 8	—	-29.445 638 951 1
2	0, 2	0d _{3/2}	25.306 694 271 9	30.000 583 808 2	27.604 868 224 2	32.297 896 695 9
2	0, 2	0d _{3/2}	-46.546 918 839 0	-62.677 905 209 6	-42.724 013 026 7	-58.604 631 382 5
3	0, 3	0f _{5/2}	34.610 196 198 4	39.161 937 688 6	36.910 768 553 0	41.467 912 838 3
3	0, 3	0f _{5/2}	-81.500 414 590 9	-103.092 083 293	-77.235 371 822 4	-98.682 629 757 0
4	0, 4	0g _{7/2}	43.672 221 634 1	48.151 824 643 7	45.984 639 418 4	50.471 098 794 0
4	0, 4	0g _{7/2}	-127.493 140 894	-154.724 921 481	-122.974 788 986	-150.123 596 323
1	1, 1	1p _{1/2}	15.694 507 311 9	20.704 785 663 5	18.008 614 664 7	23.000 238 700 3
1	1, 1	1p _{1/2}	—	-32.954 276 844 6	-42.721 748 779 4	-29.442 334 896 2
2	1, 2	1d _{3/2}	25.503 813 697 5	30.178 760 491 1	27.793 334 332 1	32.469 165 277 9
2	1, 2	1d _{3/2}	-46.544 824 826 5	-62.676 167 031 5	-77.233 682 766 6	-58.602 843 400 5
3	1, 3	1f _{5/2}	34.773 475 875 4	39.313 157 731 8	37.068 373 198 2	41.614 377 354 2
3	1, 3	1f _{5/2}	-81.498 710 376 5	-103.090 259 406	-77.233 682 766 6	-98.680 845 493 5
4	1, 4	1g _{7/2}	43.813 470 484 3	48.284 691 377 9	46.121 842 599 9	50.600 481 482 0
4	1, 4	1g _{7/2}	-127.491 139 500	-154.722 732 219	-122.972 831 579	-150.121 447 414

$$F_{n\kappa}(r) = N_{n\kappa} s^\nu (1+s)^{\eta/2+1/2} P_n^{(2\nu,\eta)}(1+2s)$$
$$\nu = \frac{d}{2} \sqrt{\delta D_0 + \gamma + \beta^2}$$
$$\eta = -\sqrt{\delta D_2 d^2 + \gamma b^2 d^2 + 1}$$

where $N_{n\kappa}$ is the normalization constant, which can be explicitly determine by using the normalization condition $\int_0^\infty |F_{n\kappa}(r)|^2 dr = 1$ [40, 41] as

$$N_{n\kappa} = \left[\frac{\Gamma(2\nu+1)}{\Gamma(n)} \sum_{p=0}^\infty \frac{(-1)^p \Gamma(n+p)(n+1+2\nu+\eta)_p \Gamma(\eta+2)}{p!(p+2\nu)\Gamma(p+2\nu+\eta+2)} \frac{1}{\alpha} {}_3F_2(p+2\nu, -n, n+2\nu+\eta+1; p+2\nu+\eta+2; 1+2\nu; 1) \right]^{-(1/2)}$$

3.2. Solutions for the pseudospin symmetry limit

In the previous section the Dirac equation satisfying the $G_{n\kappa}$ that we obtained can be transformed into a Schrödinger-like equation in the form

$$\frac{4}{d^2} \frac{d^2 F_{n\kappa}(s)}{ds^2} + \frac{4}{sd^2} \frac{dF_{n\kappa}(s)}{ds} + \frac{1}{s^2(1+s)^2} \{ -(\tilde{\delta}D_0 + \tilde{\gamma} + \tilde{\beta}^2 - \tilde{\delta}D_1 - 2\tilde{\gamma}b + \tilde{\delta}D_2 + \tilde{\gamma}b^2)s^2 + s[\tilde{\delta}(D_1 - 2D_0) + 2\tilde{\gamma}(b-1) - 2\tilde{\beta}^2] - (\tilde{\delta}D_0 + \tilde{\gamma} + \tilde{\beta}^2) \} F_{n\kappa}(s) = 0$$

Table 3. The relativistic bound state pseudospin symmetric energy eigenvalues in units of fm⁻¹ of the improved expression of the Rosen–Morse potential energy model for several values of n and κ .

$\tilde{\ell}$	$n, \kappa < 0$	ℓ, j	$C_s = 0$		$C_{ps} = -5$	
			$E_{n,\kappa<0}(T=0)$	$E_{n,\kappa<0}(A=0.5)$	$E_{n,\kappa<0}(T=0)$	$E_{n,\kappa<0}(A=0.5)$
1	1, -1	1s _{1/2}	17.129 091 094 3	11.110 553 227 4	14.893 144 9872	13.892 926 751 8
1	1, -1	1s _{1/2}	—	—	-27.683 946 698 6	—
2	1, -2	1p _{3/2}	27.983 565 555 8	22.666 937 775 5	25.738 523 073 6	25.178 737 051 5
2	1, -2	1p _{3/2}	-54.190 840 874 0	-37.747 280 710 1	-58.190 217 645 0	-34.271 172 032 3
3	1, -3	1d _{5/2}	38.281 805 393 2	33.173 198 177 9	36.014 965 502 3	35.619 895 767 5
3	1, -3	1d _{5/2}	-97.265 812 608 0	-73.985 679 472 0	-101.661 432 395	-69.845 267 303 0
4	1, -4	1f _{7/2}	48.349 368 835 3	43.335 246 419 9	46.059 688 869 1	45.761 831 675 9
4	1, -4	1f _{7/2}	-154.469 500 198	-124.086 313 701	-159.078 980 138	-119.603 439 666
1	2, -1	2s _{1/2}	17.358 578 839 1	11.391 252 188 8	15.135 910 758 3	14.139 247 925 7
1	2, -1	2s _{1/2}	—	—	-27.675 834 048 0	—
2	2, -2	2p _{3/2}	28.159 974 875 8	22.865 344 292 6	25.922 270 280 4	25.365 291 994 4
2	2, -2	2p _{3/2}	-54.187 013 248 0	-37.741 848 832 2	-58.186 569 878 5	-34.265 272 933 5
3	2, -3	2d _{5/2}	38.428 503 611 1	33.332 974 188 8	36.166 494 789 6	35.772 929 269 9
3	2, -3	2d _{5/2}	-97.262 283 388 0	-73.982 298 381 5	-101.657 827 402	-69.841 862 839 5
4	2, -4	2f _{7/2}	48.476 787 827 9	43.471 379 400 5	46.190 534 080 0	45.893 520 747 0
4	2, -4	2f _{7/2}	-154.465 167 442	-124.082 407 713	-159.074 567 553	-119.599 620 799
$\tilde{\ell}$	$n-1, \kappa > 0$	$\ell+2, j+1$	$E_{n-1,\kappa>0}(T=0)$	$E_{n-1,\kappa>0}(A=0.5)$	$E_{n-1,\kappa>0}(T=0)$	$E_{n-1,\kappa>0}(A=0.5)$
1	0, 2	0d _{3/2}	17.129 091 094 3	22.666 937 775 5	14.893 144 987 2	25.178 737 051 5
1	0, 2	0d _{3/2}	—	-37.747 280 710 1	-27.683 946 698 6	-34.271 172 032 3
2	0, 3	0f _{5/2}	27.983 565 555 8	33.173 198 177 9	25.738 523 073 6	35.619 895 767 5
2	0, 3	0f _{5/2}	-54.190 840 874 0	-73.985 679 472 0	-58.190 217 645 0	-69.845 267 303
3	0, 4	0g _{7/2}	38.281 805 393 2	43.335 246 419 9	36.014 965 502 3	45.761 831 675 9
3	0, 4	0g _{7/2}	-97.265 812 608 0	-124.086 313 701	-101.661 432 395	-119.603 439 666
4	0, 5	0h _{9/2}	48.349 368 835 3	53.334 474 261 0	46.059 688 869 1	55.755 812 666 5
4	0, 5	0h _{9/2}	-154.469 500 198	-188.424 401 929	-159.078 980 138	-183.762 762 224
1	1, 2	1d _{3/2}	17.358 578 839 1	22.865 344 292 6	15.135 910 758 3	25.365 291 994 4
1	1, 2	1d _{3/2}	—	-37.741 848 832 2	-27.675 834 048 0	-34.265 272 933 5
2	1, 3	1f _{5/2}	28.159 974 875 8	33.332 974 188 8	25.922 270 280 4	35.772 929 269 9
2	1, 3	1f _{5/2}	-54.187 013 248 0	-73.982 298 381 5	-58.186 569 878 5	-69.841 862 839 5
3	1, 4	1g _{7/2}	38.428 503 611 1	43.471 379 400 5	36.166 494 789 6	45.893 520 747
3	1, 4	1g _{7/2}	-97.262 283 388 0	-124.082 407 713	-101.657 827 402	-119.599 620 799
4	1, 5	1h _{9/2}	48.476 787 827 9	53.454 585 394 5	46.190 534 080 0	65.771 467 019 5
4	1, 5	1h _{9/2}	-154.465 167 442	-188.419 665 357	-159.074 567 553	-262.293 142 198

where we have approximated the centrifugal term and introduced the following notation so as to avoid mathematical complexity:

$$\tilde{\gamma} = (E_{n\kappa} - M - C_s)D_e \quad \beta = \sqrt{(M - E_{n\kappa} + C_s)(M + E_{n\kappa})} \quad (26)$$

and $\delta = (\kappa + T - 1)(\kappa + T)$

To avoid repetition in solutions, the negative energy solution of (18) can be readily obtained directly via the spin symmetric solution through the following parametric mappings [41, 42]:

$$\begin{aligned} F_{s,n\kappa}(r) &\leftrightarrow G_{ps,n\kappa} & \kappa &\rightarrow \kappa - 1 & V(r) &\rightarrow -V(r) \\ E_{s,n\kappa} &\rightarrow -E_{ps,n\kappa} & C_s &\rightarrow -C_{ps} \end{aligned} \quad (27)$$

By applying the preceding transformations to the equations, the following pseudospin symmetric energy equation can be found:

$$\begin{aligned} \tilde{\delta}D_0 + \tilde{\gamma} + \tilde{\beta}^2 - \tilde{\delta}D_1 - 2\tilde{\gamma}b + \tilde{\delta}D_2 + \tilde{\gamma}b^2 \\ - \frac{4}{d^2} \left(n + \frac{1}{2} - \frac{1}{2} \sqrt{\tilde{\delta}D_2 d^2 + \tilde{\gamma}b^2 d^2 + 1} \right. \\ \left. + \frac{d}{2} \sqrt{\tilde{\delta}D_0 + \tilde{\gamma} + \tilde{\beta}^2} \right)^2 = 0 \end{aligned} \quad (28)$$

and the un-normalized lower component of the spinor wave functions are expressed in terms of the Jacobi polynomials as

$$\begin{aligned} G_{n\kappa}(r) &= \tilde{N}_{n\kappa} s^{\tilde{\nu}} (1+s)^{(\tilde{\eta}/2)+(1/2)} P_n^{(2\tilde{\nu}, \tilde{\eta})}(1+2s) \\ \tilde{\nu} &= \frac{d}{2} \sqrt{\tilde{\delta}D_0 + \tilde{\gamma} + \tilde{\beta}^2} \\ \tilde{\eta} &= -\sqrt{\tilde{\delta}D_2 d^2 + \tilde{\gamma}b^2 d^2 + 1} \end{aligned} \quad (29)$$

where the normalization constant, $\tilde{N}_{n\kappa}$, is obtained as

$$\tilde{N}_{n\kappa} = \left[\frac{\Gamma(2\tilde{\nu}+1)}{\Gamma(n)} \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma(n+p)(n+1+2\tilde{\nu}+\tilde{\eta})_p \Gamma(\tilde{\eta}+2)}{p!(p+2\tilde{\nu})\Gamma(p+2\tilde{\nu}+\tilde{\eta}+2)} \frac{1}{\alpha} {}_3F_2(p+2\tilde{\nu}, -n, n+2\tilde{\nu}+\tilde{\eta}+1; p+2\tilde{\nu}+\tilde{\eta}+2; 1+2\tilde{\nu}; 1) \right]^{-1/2} \quad (30)$$

3.3. The nonrelativistic limiting case

In this section, we study energy eigenvalue (22) and the upper-spinor component of wave function (23) of the Dirac–Rosen–Morse problem under the nonrelativistic limits. Now by taking $C_s = 0$ and applying the following appropriate transformations: $M + E_{n\kappa} \rightarrow 2\mu/\hbar^2$ and $M - E_{n\kappa} \rightarrow -E_{n\ell}$ [42, 43], the nonrelativistic limit of energy equation (22) and wavefunction (23) becomes

$$E_{n\ell} = 1 + \frac{\hbar^2 \ell(\ell + 1)D_0}{2\mu D_e} - \frac{\hbar^2}{2\mu D_e} \times \left[\frac{\ell(\ell + 1)(D_1 - D_2) - (4\mu D_e b/\hbar^2)(1 + b) + (4/d^2)\zeta^2}{(4/d)\zeta} \right]^2 \quad (31)$$

where

$$\zeta = n + \frac{1}{2} - \frac{1}{2} \sqrt{\ell(\ell + 1)D_2 b^2 + \frac{2\mu D_e b^2 d^2}{\hbar^2} + 1} \quad (32)$$

Also we can get the radial wave functions of the Schrödinger equation with improved expression of the Rosen–Morse potential energy model as

$$F_{n\ell}(s) = N_{n\ell} s^{\nu_0} (1 - s)^{(1/2)(\eta_0 + 1)} P^{(2\nu_0, \eta_0)}(1 + 2s) \quad (33)$$

where

$$\nu_0 = \frac{d}{2} \sqrt{\ell(\ell + 1)D_0 + \frac{2\mu D_e}{\hbar^2} - \frac{2\mu D_e E_{n\ell}}{\hbar^2}} \quad (34)$$

$$\eta_0 = - \sqrt{\ell(\ell + 1)D_2 d^2 + \frac{2\mu D_e b^2 d^2}{\hbar^2} + 1}$$

and $N_{n\ell}$ is the normalization factor, which can be calculated from the normalization conditions.

4. Numerical results

We have given some numerical values of the energy eigenvalues in Tables 2 and 3 for various states. The following set of physical parameter values: $D_e = 15 \text{ fm}^{-1}$, $d = 5 \text{ fm}$, $M = 1 \text{ fm}^{-1}$, and $r_e = 0.8 \text{ fm}$, have been used for our numerical computation. In Table 2 we noted that the set of spin symmetry doublets, $(np_{3/2}, np_{1/2})$, $(nd_{3/2}, nd_{5/2})$, $(nf_{5/2}, nf_{7/2})$, etc., have the same energy (degenerate states). We also found that the presence of tensor interaction, say $T = 0.5$, removes the degeneracy between the states in the preceding doublets and creates a new set of spin symmetric doublets.

In Table 3, we give the numerical results for the pseudospin symmetric case. We noted the degeneracy in the following doublets: $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, etc. How the tensor interaction remove degeneracies between pseudospin doublets have also been demonstrated in the same table.

5. Conclusion

We have investigated the relativistic bound state solutions of the Dirac equation with Rosen–Morse potential energy model including Coulomb-like tensor interaction for any spin–orbit quantum number, κ . By applying an appropriate Peckeris-type approximation to deal with the spin–orbit centrifugal (pseudo-centrifugal) coupling term, we have obtained the approximate relativistic energy eigenvalue equation and the normalized two components of the wave functions are expressed in terms of the Jacobi polynomials. We found that tensor interaction removes degeneracies between each pair of pseudospin or spin doublets. A

straightforward extension to the nonrelativistic limits of the problem have also been considered. The result obtained in this study can be applied in both nuclear and hadron physics.

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Appendix A. The Nikiforov–Uvarov method

One of the calculational tools used in solving the Schrödinger-like equation including the centrifugal barrier and (or) the spin-orbit coupling term is the Nikiforov–Uvarov method. This method was proposed to solve the hypergeometric-type second-order differential wave equation

$$\Psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\Psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\Psi(s) = 0 \quad (\text{A1})$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second degree, and $\tilde{\tau}(s)$ is a polynomial of first degree [A1–A3]. Prime denotes differentiation with respect to s . To make the application of the Nikiforov–Uvarov method simpler and direct, without the need to check the validity of the solution, a shortcut to the method has been presented [A4]. Firstly, it is necessary to write Schrödinger-like equation (A1) in a more general form applicable to any potential of interest as follows [A4]:

$$\Psi''_n(s) + \frac{a_1 - a_2s}{s(1 - a_3s)}\Psi'_n(s) + \frac{-\zeta_1s^2 + \zeta_2s - \zeta_3}{s^2(1 - a_3s)^2}\Psi_n(s) = 0 \quad (\text{A2})$$

satisfying the wavefunctions

$$\Psi_n(s) = \phi(s)y_n(s) \quad (\text{A3})$$

By comparing (A2) with its counterpart (A1), the following identifications can be found:

$$\tilde{\tau}(s) = a_1 - a_2s \quad \sigma(s) = s(1 - a_3s) \quad \tilde{\sigma}(s) = -\zeta_1s^2 + \zeta_2s - \zeta_3 \quad (\text{A4})$$

By following the Nikiforov–Uvarov method [A2], the following parametric constants can be obtained:

$$\begin{aligned} a_4 &= \frac{1}{2}(1 - a_1) & a_5 &= \frac{1}{2}(a_2 - 2a_3) & a_6 &= a_5^2 + \zeta_1 \\ a_7 &= 2a_4a_5 - \zeta_2 & a_8 &= a_4^2 + \zeta_3 & a_9 &= a_3(a_7 + a_3a_8) + a_6 \\ a_{10} &= a_1 + 2a_4 + 2\sqrt{a_8} & a_{11} &= a_2 - 2a_5 + 2(\sqrt{a_9} + a_3\sqrt{a_8}) \\ a_{12} &= a_4 + \sqrt{a_8} & a_{13} &= a_5 - (\sqrt{a_9} + a_3\sqrt{a_8}) \end{aligned} \quad (\text{A5})$$

The essential polynomial functions are

$$\begin{aligned} \pi(s) &= a_4 + a_5s - [(\sqrt{a_9} + a_3\sqrt{a_8})s - \sqrt{a_8}] \\ k &= -(a_7 + 2a_3a_8) - 2\sqrt{a_8a_9} \\ \tau(s) &= a_1 + 2a_4 - (a_2 - 2a_5)s - 2[(\sqrt{a_9} + a_3\sqrt{a_8})s - \sqrt{a_8}] \\ \tau'(s) &= -2a_3 - 2(\sqrt{a_9} + a_3\sqrt{a_8}) < 0 \end{aligned} \quad (\text{A6})$$

The energy equation can be found as

$$\begin{aligned} a_2n - (2n + 1)a_5 + (2n + 1)(\sqrt{a_9} + a_3\sqrt{a_8}) \\ + n(n - 1)a_3 + a_7 + 2a_3a_8 + 2\sqrt{a_8a_9} = 0 \end{aligned} \quad (\text{A7})$$

and the eigenfunction as

$$\Psi_n(s) = N_n s^{a_{12}} (1 - a_3s)^{-a_{12} - (a_{13}/a_3)} P_n^{(a_{10}-1, (a_{13}/a_3) - a_{10}-1)}(1 - 2a_3s) \quad (\text{A8})$$

where P_n is the orthogonal Jacobi polynomial with

$$P_n^{(x,y)}(1 - 2a_3s) = \frac{(x + 1)_n}{n!} {}_2F_1(-n, 1 + x + y + n; x + 1, a_3s) \quad (\text{A9})$$

and N_n is a normalization constant. The preceding wavefunctions can also be expressed in terms of the hypergeometric function as

$$\begin{aligned} \Psi_n(s) &= N_n s^{a_{12}} (1 - a_3s)^{-a_{12} - (a_{13}/a_3)} P_n^{(a_{10}-1, (a_{13}/a_3) - a_{10}-1)}(1 - 2a_3s) \\ &\quad \times {}_2F_1(-n, (a_{13}/a_3) - 1 + n; a_{10}, a_3s) \end{aligned} \quad (\text{A10})$$

where $a_{12} > 0$, $a_{13} > 0$ and $s \in [0, 1/a_3]$, $a_3 \neq 0$.

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