

Impact of Radiation Pressure on the Out-Of-Plane Libration Points in The Restricted Three-Body Problem with Rotating Mass Dipole Effect

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Abstract

This paper investigates the impact of radiation pressure of the primary body on a third body in the vicinity of the out-of-plane libration points of the restricted problem of three bodies under a rotating mass dipole effect. It studies the impact of radiation factor (q_1), force ratio (k) and mass-ratio parameter (μ) on the location, zero-velocity curves and instability of the out-of-plane libration points in the frame work of the CRTBP. A symmetrical out-of-plane libration points are identified numerically on the xz – plane and the three system's parameters, that is q_1, k and μ , are found to have impact on the positions of these points. In addition, the force ratio and radiation factor have significant effect on the configuration of the zero-velocity contours. The linear stability of the points for a broad scope of the parameters' combinations was investigated and it is found that the out-of-plane libration points are generally points of instability.

Keywords: CRTBP; Rotating mass dipole; Radiation pressure; Out-of-plane libration points; Zero velocity curves.

INTRODUCTION

The restricted three-body problem (RTBP) is concerned with the motion of a test body of infinitesimal mass which is attracted by a pair of point masses, termed primaries, which move in elliptical or circular paths about their barycenter of mass. The test body (also known as third body) has no effect on the motions of the primary bodies. If the primaries are confined to circular orbit while the third body moves in the plane of motion prescribed by the two primaries, in that case, it is known as circular RTBP (CRTBP). The CRTBP admits five libration points, labeled $L_i, i = 1, 2, \dots, 5$ where $L_{1,2,3}$ are the collinear libration points, which are always unstable, while L_4 and L_5 called triangular libration points are in equilateral triangular

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configuration with the two primaries, and are stable for $\mu \in (0, \mu_c)$ where μ is the mass parameter and $\mu_c = 0.03852$ is the critical mass parameter (Murray and Dermott, 1999).

A dipole-model comprises of the two primary bodies of the CRTBP firmly contacting one another by a massless rod in a constant characteristic distance. The motion of the test body or third body is restricted to the orbital plane of the primary bodies.

On some occasions, the classical version of CRTBP model is inadequate to appropriately portray the real cases; a few modifications have to be initiated, and their impacts considered. One modification consists of considering either one or both primaries as radiation sources. This model is called the photogravitational restricted three-body problem (Radzievskii, 1950; Perdios *et al.*, 2008) and has attracted much interest (Kalvouridis *et al.*, 2006; Leke *et al.*, 2025; Papadouris & Papadakis, 2013; Singh & Vincent, 2015; Joshi & Hasan, 2023). The existence of the out-of-plane libration points was first pointed out by Radzievskii (1953) in the photogravitational restricted three-body problem. The stability of these points was first studied in the solar problem (Sun-Jupiter-particle) by Perezhogin (1976) in the whole range of existence when only the dominant body radiates. They demonstrated that the two out-of-plane libration solutions are generally unstable. This work was extended by Schuerman (1980) who considered both primary bodies as sources of radiation, and the possibility of a radiation force greater than gravitational attraction. Later, Simmons *et al.*, (1985) made a complete analysis of the positions and stability of the equilibria (both planar and out-of-plane) for all values of the radiation factors and the mass parameter. Other authors (Ragos and Zagouras; Ragos and Zagouras, 1993; Xuetao and Lizhong, 1993; Das *et al.*, 2009; Kalantonis *et al.*, 2021; Idrisi and Ullah, 2024; Moneer *et al.*, 2023; Vincent *et al.*, 2024b; Vincent *et al.*, 2024c; Jat & Kishor, 2025) continued in this line to study the libration points of out of plane when some radiation parameters are negative.

Besides the above intensive studies on the modifications of the CRTBP and its variants, its generalized form of the rotating mass dipole has also been presented by Chermnykh (1987) to approximate the rotating dumbbell. In this classical rotating mass dipole model, the common force of gravity uniting the primary bodies may be above or under the force of centrifugal. Later, Godziewski and Maciejewski (1999); Gozdziewski (2003); Papadakis (2005); Perdios *et al.*, (2015) among others, made an extensive study on the rotating dipole system with a point mass and a rigid body. Prieto-Llanos and Gmez-Tierno (1994) revisited the dipole model (two-fixed-centers) system with elongated bodies in rotation focusing on the libration solutions and their stability under the condition gravitational force exceeds the centrifugal force. Later, Hirabayashi *et al.* (2010) focused on the collinear libration points in the case where centrifugal force exceeds the gravitational one. Various modifications of the classical rotating mass dipole system have been studied to comprehend the impact of diverse parameters in practicable celestial systems (Wang *et al.*, 2014; Zeng *et al.*, 2015; Zeng *et al.*, 2016a, b, c; Agbogidi *et al.*, 2025).

Here, the work is an extension of Prieto-Llanos and Gmez-Tierno (1994) by investigating the out-of-plane libration points when the primary body is a source radiation. The goal of this work is to numerically examine the out-of-plane libration points of the modified rotating mass dipole model and their stability properties, together with the areas of feasible motion given by the zero-velocity curves. The studies previously mentioned on the generalized rotating mass dipole model, no research has been carried out along the direction of out-of-plane libration points with allowance for radiation pressure force. Hence, current work is on investigating the motion of the third body having an infinitesimal mass near the out-of-plane

libration solutions under the influence of mass parameter, radiation pressure factor and force ratio parameter.

METHODOLOGY

Mathematical formulations

Denote m_1 and m_2 ($m_2 < m_1$) as the masses primaries of the rotating mass dipole system, connected on the x – axis by a massless rod (we have considered m_2 to be on the positive x – axis) and m the mass of the infinitesimal third body. The characteristic distance from m_1 to m_2 is kept constant value of d regardless of the rotation period. The motion of primaries m_1 and m_2 comprises of round orbits about their center of mass; whereas the infinitesimal mass m of the third body is significantly small and does not influence the motion of the primaries. The unit of mass is set as $M(=m_1+m_2)$, the unit of length is defined as the characteristic distance d and time unit is taken as ω^{-1} , where $\omega = \omega z$ is angular velocity oriented with the major axis possessing high moment of inertia, making 2π the periodic time of rotation of the primaries. Let the distances of m from m_1, m_2 and origin (O) of the system be r_1, r_2 and r respectively. Defining the mass parameter μ of the system as $\mu = \frac{m_2}{m_2+m_1} \leq \frac{1}{2}$ with $m_2 = \mu$ and $m_1 = 1 - \mu$. The coordinates of m, m_2 and m_1 in the rotating coordinate system (oxy) are (x, y, z) , $(1 - \mu, 0, 0)$ and $(-\mu, 0, 0)$ correspondingly see Fig. 1.

The equations of motion of the infinitesimal body in the gravitational field of the modified rotating mass dipole are given by Ragos and Zagouras (1993); Prieto-Llanos and Gmez-Tierno (1994).

$$\begin{aligned}\ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z,\end{aligned}\tag{1}$$

where dots denote differentiations with respect to time t and Ω is the potential force function acting on the infinitesimal mass defined as

$$\Omega = \frac{(x^2 + y^2)}{2} + k \left(\frac{q_1(1-\mu)}{r_1} + \frac{\mu}{r_2} \right),\tag{2} \text{ with}$$

$$r_1 = \left| \sqrt{(x+\mu)^2 + y^2 + z^2} \right|, \quad r_2 = \left| \sqrt{(x+\mu-1)^2 + y^2 + z^2} \right|\tag{3}$$

and $k = \frac{GM}{\omega^2 d^3}$ is the rotation parameter also known as the force ratio, G is the gravitational constant with a numerical value $G = 6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$. The force ratio $k \in (0, +\infty)$ and if $k = 1$ (classical CRTBP), m_1 and m_2 orbit their center of mass with no internal forces in the dipole system. In case when $k < 1$ the dipole is stretching and when $k > 1$, it is compressing (Zeng *et al.*, 2016a;2016b). Furthermore, in the present study the values of force ratio will mainly lie in the interval $k \in (0, 9]$, without excluding analysis for higher values of k . q_1 is the radiation factor of the primary body. The radiation force effect according to Radzievskii (1953)

is given by the relation $q = 1 - b$, where $b = \frac{F_r}{F_g}$, F_r is the force produced by the radiation and F_g is the gravitational force due to the radiating mass m_1 . Note that if $q_1 = 1 (\Rightarrow b = 0)$ and $k = 1$ the work reduces to that of classical RTBP, which considers only the effect of gravity. If $q_1 \in (0, 1) \Rightarrow b \in (0, 1)$, then the radiation force is less than gravitational force. Also, if $q_1 < 1$, i.e., $b > 1$, force of gravitational attraction does not exceed the radiation pressure force. For extremely small dust particles or spacecraft there is the possibility for the value of q_1 to be negative, such circumstances are incorporated in this present study (i.e. modified rotating mass dipole model). Finally, if $q_1 = 0 \Rightarrow b = 1$, gravitational and radiation forces are equal. Thus, the values of q_1 in the present problem could be taken in interval $(-\infty, 1]$.

The Jacobian integral of this problem is

$$(x^2 + y^2) + 2k \left(\frac{q_1(1-\mu)}{r_1} + \frac{\mu}{r_2} \right) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = C \quad (4)$$

where C is known as Jacobi constant of integration.

This problem is work out by mechanism of three free parameters: force ratio, k , mass parameter, μ , and radiation factor, q_1 . Contrary to the classical RTBP where the zero-velocity curves are chiefly derived by the mass parameter, the zero-velocity curves of this modified rotating mass dipole are determined by all the parameters. It is worth noticing that in case where $q_1 = 1$, the problem reduces to that of Prieto-Llanos and Gmez-Tierno (1994) while we get the classical RTBP case when $q_1 = k = 1$ (Murray and Dermott, 1999).

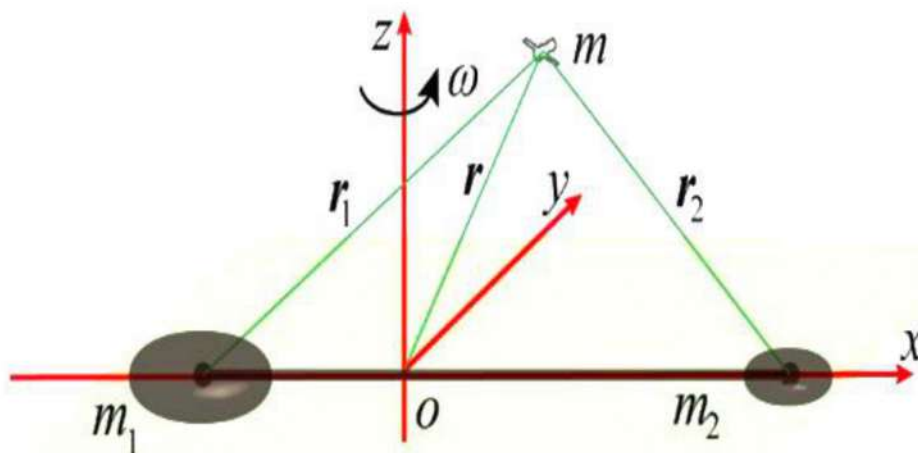


Fig. 1. The geometric representation of the synodic coordinate system (Zeng *et al.*, 2016a).

Results

Location of the out-of-plane libration points

By utilizing analytical and numerical approaches to study the presence and position of the out-of-plane libration points of the rotating mass dipole with non-zero tensile force provided by the rod. The respective coordinates (x, y, z) of the libration points are determined using Equation (1) by putting velocity $(\dot{x}, \dot{y}, \dot{z})$ and acceleration $(\ddot{x}, \ddot{y}, \ddot{z})$ components equal to zero and solving the resulting system,

$$\Omega_x = 0, \Omega_y = 0, \Omega_z = 0 \quad (5)$$

for x, y and z where $\Omega_x, \Omega_y, \Omega_z$ are, respectively

$$\begin{aligned} x - k \left(\frac{q_1(1-\mu)(x+\mu)}{r_1^3} + \frac{\mu(x+\mu-1)}{r_2^3} \right), \\ y \left(1 - k \left[\frac{q_1(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} \right] \right) \quad \text{and} \\ -zk \left(\frac{q_1(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} \right). \end{aligned} \quad (6)$$

In the classical rotating mass dipole problem, that is, when $q_1 = 1$, all the five libration points are found on the (x, y) -plane with $z = 0$. Since we are mainly interested in the “out-of-plane” libration points of the system, we focus on the third equation of equations (5), i.e.

$$-zk \left(\frac{q_1(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} \right) = 0.$$

This implies

$$\frac{q_1(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} = 0, \quad (7)$$

since k is always positive and $z \neq 0$, thereby making the second equation of (5) to have $y = 0$. It is clear that in the classical case, $q_1 = 1$ and the relation (7) is never satisfied.

Now, equation (7) can be expressed as

$$\frac{r_1}{r_2} = \left(-\frac{q_1(1-\mu)}{\mu} \right)^{\frac{1}{3}} \equiv \kappa_0, \quad (8)$$

which signifies κ_0 is a constant; and hence the locus of these points, form a circle of Apollonius. Now, for equation (8) to have any real solution $(x_0, 0, z_0)$, either

$$q_1 < 0 \quad \text{or} \quad q_1 = 0. \quad (9)$$

The case $q_1 = 0$ will not be considered in this model problem. Thus, for the out-of-plane

libration points, the first and the last equations of (5) must be solved when $y = 0$, since the points lay on the plane of Oxz . That is, the points arise from the solution of the system

$$x_0 - k \left(\frac{q_1(1-\mu)(x_0 + \mu)}{r_{10}^3} + \frac{\mu(x_0 + \mu - 1)}{r_{20}^3} \right) = 0, \quad (10)$$

$$\frac{q_1(1-\mu)}{r_{10}^3} + \frac{\mu}{r_{20}^3} = 0,$$

with

$$r_{10}^2 = (x_0 + \mu)^2 + z_0^2, \quad r_{20}^2 = (x_0 + \mu - 1)^2 + z_0^2 \quad (11)$$

The presence and positions of these libration points rely on μ , q_1 and k .

Now, utilizing the first equation of equations (10) with the equation (11), we get x_0 - coordinate of the out-of-plane libration points as

$$x_0 = \frac{1}{2} - \mu + \frac{1}{2} r_{20}^2 (\kappa_0^2 - 1) \quad (12)$$

Using equation (12) in equations (10), we obtain that the distance of any real out-of-plane libration point from m_2 must satisfy the polynomial equation

$$P(r_{20}) = a_2 r_{20}^5 + a_1 r_{20}^3 + a_0 = 0, \quad (13)$$

with

$$a_2 = (1 - \kappa_0^2), \quad a_1 = (2\mu - 1), \quad a_0 = -2k\mu$$

Equation (13) shows that the distances of the out-of-plane libration points with respect to the smaller primary m_2 are functions μ, k , and q_1 . We remark that any positive root jointly with the consistent value of r_{10} acquired from equations (10) have to satisfy the inequality stated as $r_{20} + r_{10} > 1$, otherwise, the out-of-plane libration points will vanish. According to Descartes' rule of signs, if there are no real positive roots of (13), then there are no out-of-plane libration points; if there only one real positive root of equation (13), then there exist at most one pair of libration points, namely $L_{1,2}^z$, while they always satisfy the above condition.

Substituting equation (12) in second of equations (11) in combination with equation (8), we have

$$z_0 = \pm \sqrt{r_{20}^2 - \frac{1}{4} \left\{ 1 + (1 - \kappa_0^2) r_{20}^2 \right\}^2}, \quad \left(\kappa_0 = \left(-\frac{q_1(1-\mu)}{\mu} \right)^{\frac{1}{3}} \right) \quad (14)$$

From equations (12) and (14), we can locate the coordinates of the out-of-plane libration points

$L_{1,2}^z(x_0, 0, \pm z_0)$ in the CRTBP with rotating mass dipole under radiating primary. Note that for some certain range of values of μ, q_1, k , the libration points may disappear through bifurcation. To this end, the effects of mass parameter, force ratio and radiation pressure of the massive primary body on the locations of $L_{1,2}^z(x_0, 0, \pm z_0)$ are graphically and numerically presented. We consider individual effect of μ, k and q_1 on the positions of the libration points. It is worth highlighting that these libration points exist only if the parameter $q_1 < 0$ holds. In case of $q_1 = 0, k = 1$, solving equations (12) and (14) give three solutions on the x – axis which are real. Two of the solutions are coordinates of the two primaries, while the other one is a collinear point. Consequently, in view of the evaluation rooted on the inequalities (9), we wrap up that for $q_1 > 0$ there are no libration points on the Oxz plane (Radzievskii, 1953).

The numerical results of the exact locations of the equilibria have been obtained by Newton–Raphson algorithm (Gousidou-Koutita and Kalvouridis 2009) because it is fast and always converges. By applying this method, we obtain the positions of $L_{1,2}^z(x_0, 0, \pm z_0)$ through the contour of surfaces $\Omega_x(x, 0, z, \mu, k, q_1) = 0$ (line colored red) and $\Omega_z(x, 0, z, \mu, k, q_1) = 0$ (line colored blue). In Fig.2, we show these curves for the values $\mu = 0.08, k = 1.2$, and $q_1 = -0.01$. It is noteworthy the collinear point L_c which appears as libration point in Fig.2. We find that there are only two symmetric libration points on the Oxz – plane. Specifically, L_1^z when $z > 0$ and its symmetric point in terms of the Oxy plane and L_2^z when $z < 0$. Apparently, these numerical results validate the analytical conclusion drawn above. Furthermore, when $q_1 < 0$, there are no triangular points, while collinear libration point does exist depends on the problem parameters μ, k , and q_1 (Xuetang and Lizhong, 1993). Obviously, the Ox libration point corresponds to the one of the outer collinear libration points found in the classical planar rotating dipole problem (Prieto-Llanos and Gmez-Tierno, 1994). However, we are only concerned here with $L_{1,2}^z(x_0, 0, \pm z_0)$, such a configuration of relative Lagrangian libration where exist are omitted from this study.

In order to show the effects of μ, k and q_1 on $L_{1,2}^z(x_0, 0, \pm z_0)$, the investigated regions for the radiation factor, mass parameter and force ratio are taken to be $-1 < q_1 < 0, 0 < \mu \leq 0.5$ and $0 < k \leq 9$, respectively. The respective coordinates are obtained in Table 1 for a case of only the k varying in the interval $k \in [0.18, 9.0]$ for fixed values of $q_1 = -0.055$ and $\mu = 0.5$. It is noted that increase in k results to increase in the z – coordinates of the libration points in the interval $0.18 \leq k \leq 3.0$ and decrease in the interval $3.0 < k \leq 9.0$, while the x – coordinates constantly increase from the origin. This effect is shown in Fig. 3. It is noted that the locations of the symmetrical libration points $(x_0, \pm z_0)$ shift away from the location m_1 and after attaining maximum in z_0 they descend back in anticlockwise direction to the line connecting the primaries, i.e. Ox – axis. The discrepancy tendency of the libration locations with other specified values of μ is analogous to the situation with $\mu = 0.5$, especially when the parameter k is pushed at its limits. The green circles bold marker denote the coordinates of the libration points along with the variation of k . L_1^z and L_2^z are symmetric a such they possess identical behavior hence, we have used the numbers (1, 2, ..., 9) for curves of L_1^z . Table 2 shows

the coordinates of the libration points L_1^z and L_2^z obtained numerically for $q_1 = -0.055$, $k = 1.5$ with respect to various values of $\mu \in (0, 0.5]$. It is observed that with increase in the values of μ , for fixed $q_1 = -0.055$, $k = 1.5$, the z – coordinates of the L_1^z and L_2^z reduce while the x – coordinates constantly shift away from the origin. Fig. 4 depicts this variational tendency. The green circles bold marker denotes the coordinates of the libration points, while we have used the numbers (1, 2, ..., 9) for each case of mass parameter μ corresponding to the L_1^z and L_2^z . Only the sketch map of the primaries corresponding to the case $\mu = 0.1$ are given.

In a similar manner, for the examination of the effect of q_1 on the locations of L_1^z and L_2^z we fix $\mu = 0.075$ and $k = 1.5$. For increasing values of q_1 , the coordinates of the libration points are presented in Table 3. It is observed that as q_1 increases the z – coordinates of the libration points draw closer to the line joining the primaries, while the x – coordinates increase away from the center of mass. This effect is depicted in Fig.5, from the plots, as the radiation factor q_1 varies, the locations of L_1^z and L_2^z draw toward the Ox – axis, implying that there is a shift in both x and z – coordinates. It is important to note that the positions of L_1^z and L_2^z change in a comparatively little scope with increase in radiation pressure as the mass parameter increases. Observe the distinct values of the z – coordinates in Figs.3-5.

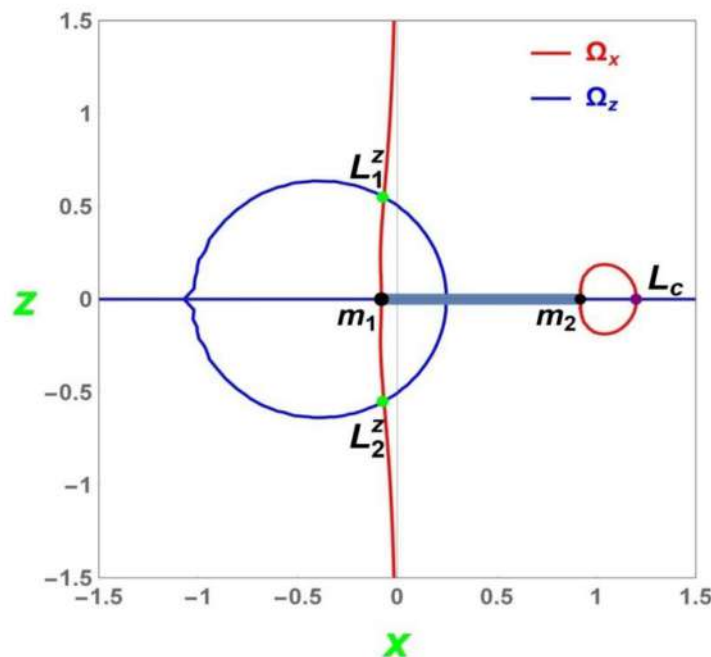


Fig.2. The positions L_1^z and L_2^z through the crossways of $\Omega_x = 0$ (curve red) and $\Omega_z = 0$ (curve blue), when $\mu = 0.08$, $k = 1.2$, $q_1 = -0.01$. The green spots signify the positions of the libration points L_1^z and L_2^z , while black spots indicate the locations of m_1 and m_2 . The purple dot (L_c) is the collinear point where exists.

Table 1. The coordinates of $L_{1,2}^z(x_0, \pm z_0)$ for distinct values of k

q_1	μ	k	x_0	$\pm z_0$
-0.055	0.5	0.18	-0.229284	0.065161
-0.055	0.5	1.00	-0.455265	0.389806
-0.055	0.5	2.00	-0.600726	0.439309
-0.055	0.5	3.00	-0.706501	0.443017
-0.055	0.5	4.00	-0.792662	0.427073
-0.055	0.5	5.00	-0.866667	0.398275
-0.055	0.5	6.00	-0.932233	0.358348
-0.055	0.5	7.00	-0.991524	0.306092
-0.055	0.5	8.00	-1.045920	0.235904
-0.055	0.5	9.00	-1.096380	0.122787

Table 2. The coordinates of $L_{1,2}^z(x_0, \pm z_0)$ for different values of μ

q_1	k	μ	x_0	$\pm z_0$
-0.055	1.5	0.10	-0.041403	1.213530
-0.055	1.5	0.15	-0.103465	0.877169
-0.055	1.5	0.20	-0.167910	0.731822
-0.055	1.5	0.25	-0.231783	0.643631
-0.055	1.5	0.30	-0.294525	0.580930
-0.055	1.5	0.35	-0.356129	0.531975
-0.055	1.5	0.40	-0.416720	0.491288
-0.055	1.5	0.45	-0.476441	0.455904
-0.055	1.5	0.50	-0.535427	0.424031

Table 3. The coordinates of $L_{1,2}^z(x_0, \pm z_0)$ for different values of q_1

k	μ	q_1	x_0	$\pm z_0$
1.5	0.075	-0.055	-0.014843	1.724910
1.5	0.075	-0.050	-0.019415	1.527820
1.5	0.075	-0.045	-0.024420	1.366470
1.5	0.075	-0.040	-0.029852	1.228790
1.5	0.075	-0.035	-0.035718	1.107130
1.5	0.075	-0.030	-0.042040	0.996150
1.5	0.075	-0.025	-0.048869	0.891646
1.5	0.075	-0.020	-0.056291	0.789743
1.5	0.075	-0.015	-0.064461	0.685946
1.5	0.075	-0.010	-0.073687	0.573166

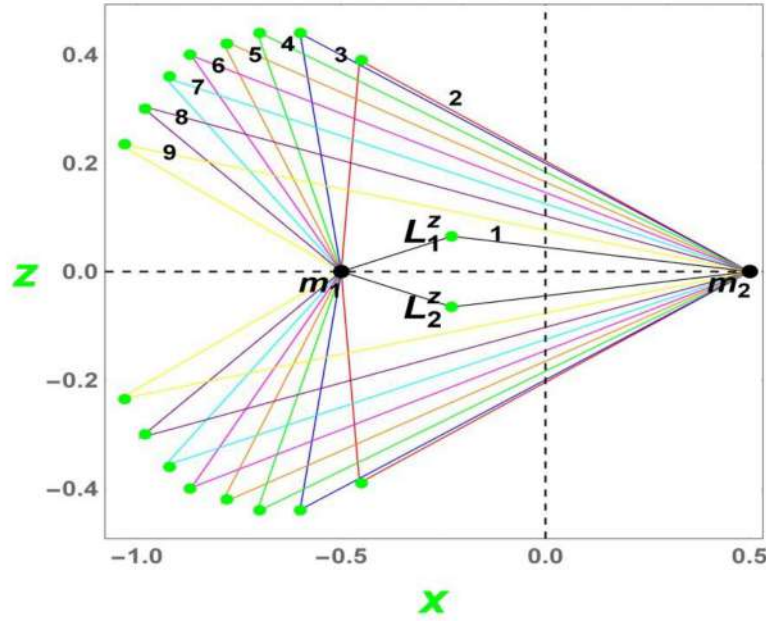


Fig.3. The locations of $L_{1,2}^z$ with $\mu = 0.5$ and $q_1 = -0.055$ when k varies.

Green dots denote the positions of $L_{1,2}^z$ while the numbers 1,2, 3,...,9 indicate curves for $k = 0.18, 1, 2, 3, \dots, 8$ correspondingly. Black dots denote the centers of the primary bodies m_1 and m_2

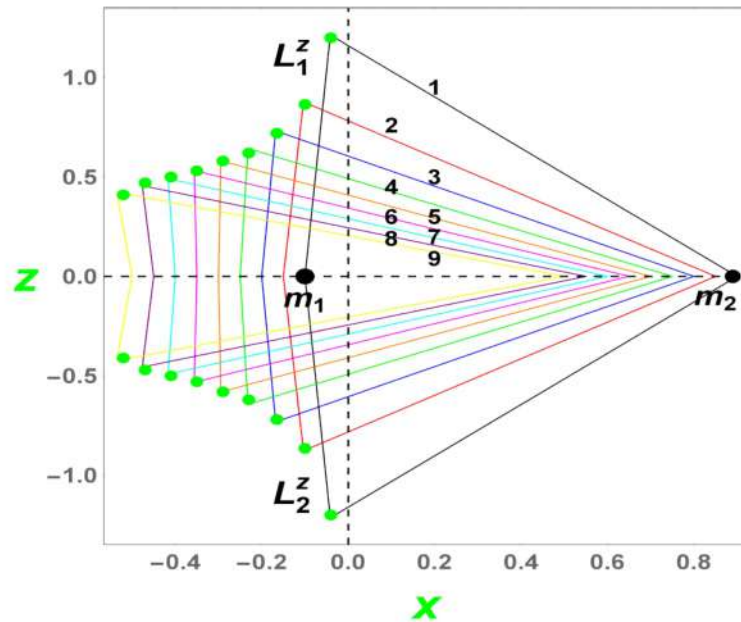


Fig.4. The positions of $L_{1,2}^z$ for constant values of $q_1 = -0.055$ and $k = 1.5$ when the mass parameter μ varies. Green dots denote the positions of $L_{1,2}^z$ while the numbers 1,2, 3,...,9 indicate curves for $\mu = 0.1, \mu = 0.15, \mu = 0.2, \mu = 0.25, \mu = 0.3, \mu = 0.35, \mu = 0.4, \mu = 0.45$ and $\mu = 0.5$, respectively. Black dots represent the centers of m_1 and m_2 .

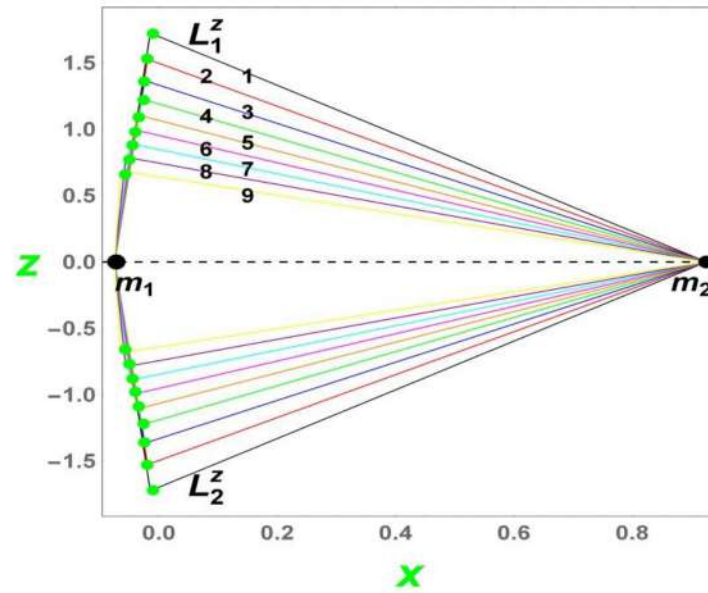


Fig.5. The positions $L_{1,2}^z$ for constant values of $\mu = 0.075$ and $k = 1.5$ when the radiation factor q_1 varies. Green dots denote the positions of while the numbers 1,2, 3...,9 indicate curves for $q_1 = -0.055, -0.050, -0.045, \dots, -0.015$ respectively. Black dots represent the centers of m_1 and m_2 .

The zero-velocity curves in the xz -plane

This section is devoted to the plots of zero velocity curves (ZVCs) using equation (4) on the (x, z) plane. The projection of surfaces is the region of possible motion for a given Jacobian constant value C in the field of two-mass rotating dipole system with involved perturbing parameters. In Fig.6, we present ZVCs for fixed values of $k = 1.5$, $\mu = 0.25$ when the values of q_1 vary, i.e., $q_1 = -0.09, q_1 = -0.04$ and $q_1 = -0.005$. The plot for the curves (thick green lines) were made only for the values of C corresponding to the libration point L_1^z (or L_2^z). Black dots indicate the primaries m_1 and m_2 , red dots are the libration points L_1^z and L_2^z of the system, shaded color gray relates to regions of impossible motion while white areas show the permissible regions where motion can occur agreeing to the constant C . The ZVCs are presented (frames a to b) in increasing order of values of C . Note that the Jacobian constant at points L_1^z and L_2^z takes same values ($CI_1^z = CI_2^z$), hence equilibria L_1^z and L_2^z are on same zero velocity curves. From this plot, near primary m_1 and through the libration points L_1^z and L_2^z , the zero-velocity curves form little ovate shape regions which do not allow motion and diminish with increase in q_1 , agreeing with Kalantonis *et al.*, (2021) in the case of zero tensile force provided by the rod. Similarly, the ZVCs for fixed values of $\mu = 0.35, q_1 = -0.08$, and for three cases of rotating mass dipole, $k = 0.2, k = 1$, and $k = 9$ are illustrated in Fig. 7. It is seen that in-between L_1^z and L_2^z , the ZVCs form little ovals of areas (where m is trapped and cannot move round m_1) which increase in size with increasing the rotating mass dipole and Jacobian constant values. It means that the parameter k has remarkable influence on the topological configuration of the areas permitted to motion of the third body m as established

by the zero-velocity tops and the correlated equipotential contours.

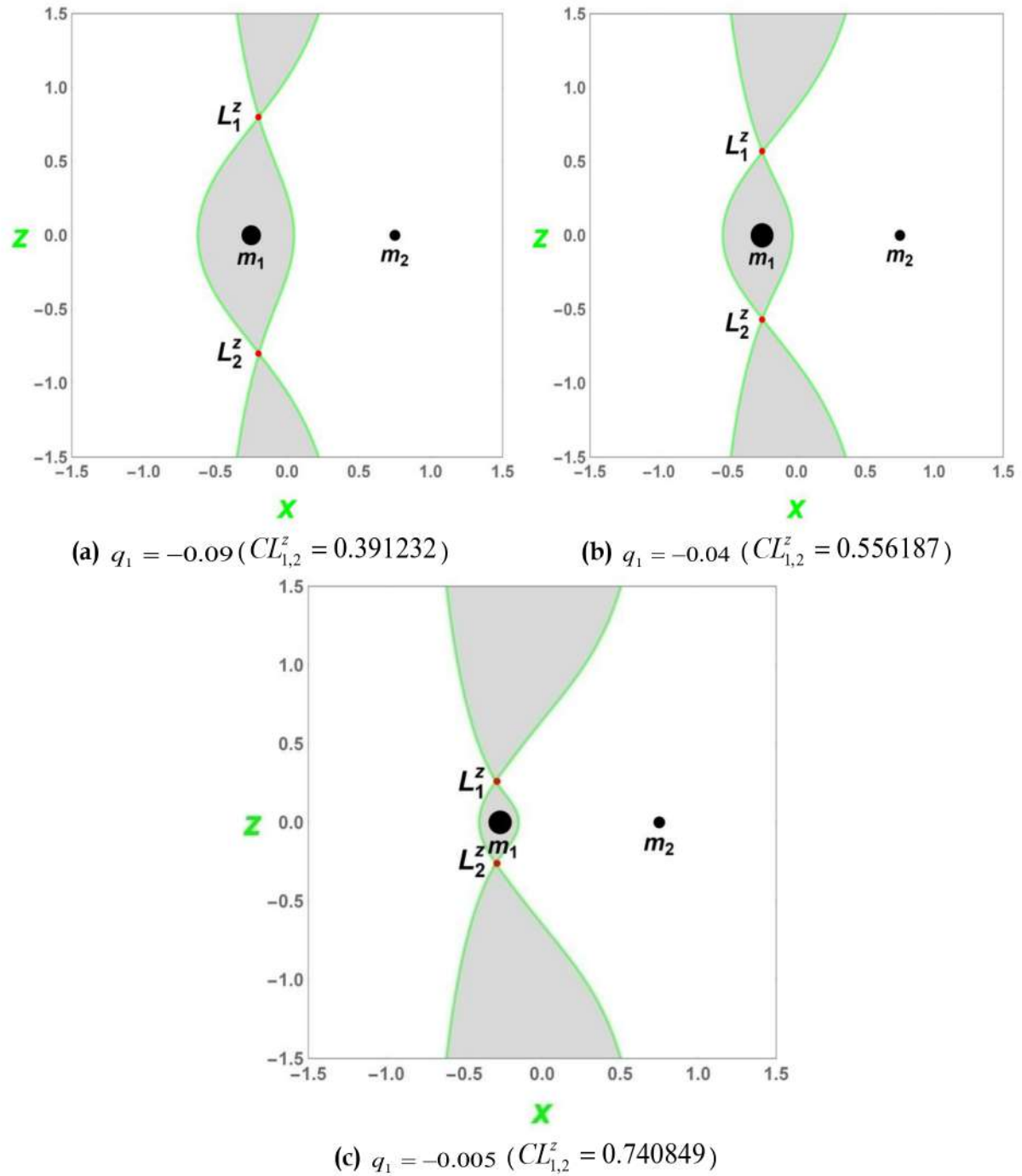


Fig.6. The Zero velocity curves $C = 2\Omega(x, z)$ and positions of L_1^z and L_2^z , for three different values of radiation factor with corresponding Jacobian constant when $\mu = 0.25$ and $k = 1.5$ are fixed. The gray region is non-reachable to motion, owing to the integral of Jacob. Notice how restricted out-of-plane motion becomes near the massive primary body m_1 .

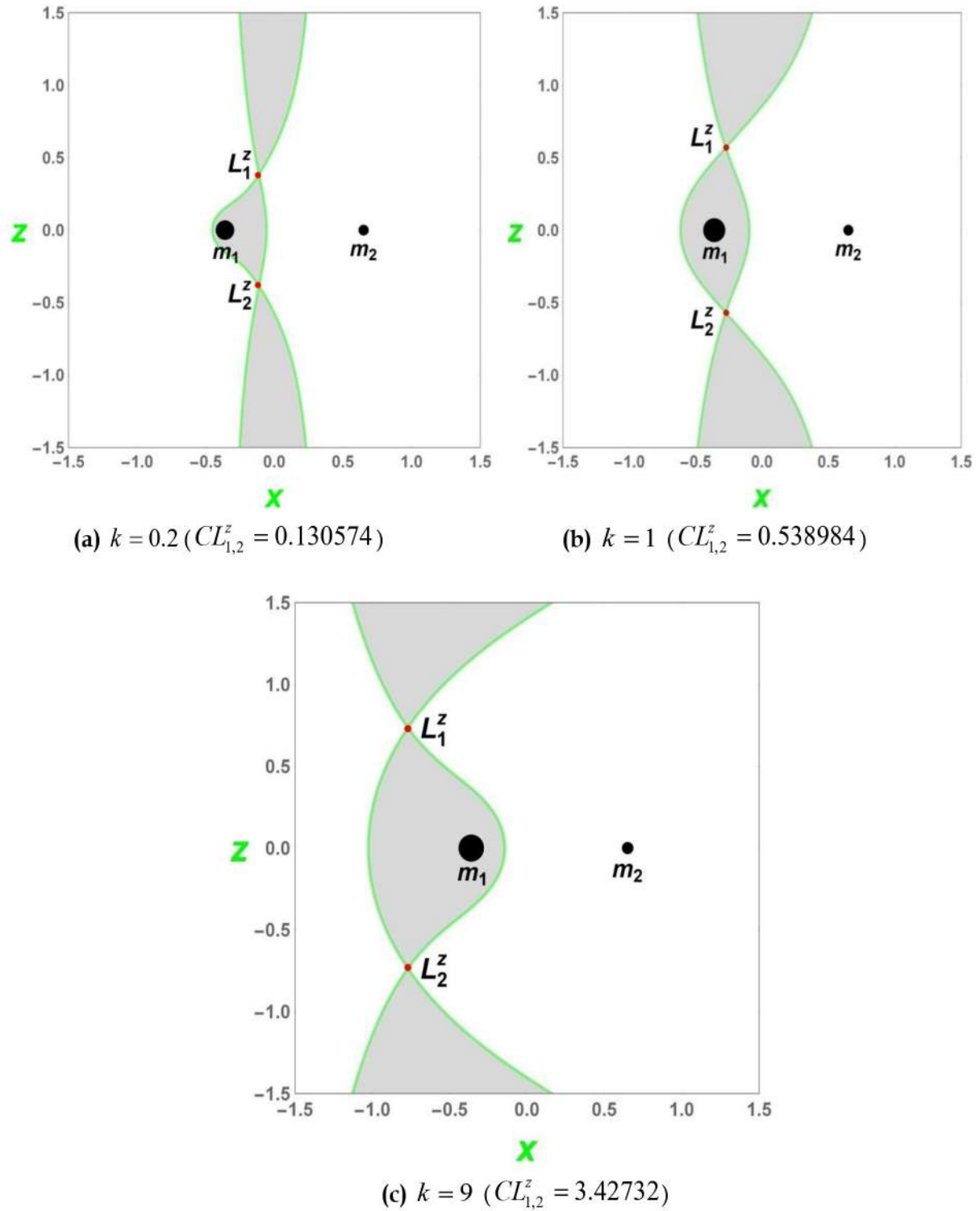


Fig.7. The Zero velocity curves $C = 2\Omega(x, z)$ and positions of L_1^z and L_2^z , for three different cases of rotating mass dipole with corresponding Jacobian constant when $\mu = 0.35$ and $q_1 = -0.08$ are fixed. The gray region is non-reachable to motion, owing to the integral of Jacobi. Notice how restricted out-of-plane motion becomes near the massive primary body m_1

RESULTS DISCUSSION

Stability analysis of the libration points

To investigate the stability in linear sense of motion in the neighborhood of any of the libration points $L_{1,2}^z(x_0, 0, \pm z_0)$ we adopt a small displacements ξ_1, ξ_2 and ξ_3 such that

$$x = x_0 + \xi_1, y = \xi_2, z = z_0 + \xi_3 \quad (15)$$

Substituting equations (15) into equations (1) and performing a series expansion due to Taylor considering first-order only, the linearized of equations for infinitesimal motion in the vicinity of the libration points in the photogravitational R3BP with rotating dipole can be written as

$$\dot{\vec{\xi}} = B \vec{\xi}, \quad \vec{\xi} = (\xi_1, \xi_2, \xi_3, \dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3)^T \quad (16)$$

where $\vec{\xi}$ is the vector's state of m in relation to $L_{1,2}^z$ and B is a time nondependent coefficient matrix stated as

$$B = \begin{pmatrix} \mathbf{0}_{3 \times 3} & I_{3 \times 3} \\ \Omega_{3 \times 3} & \chi_{3 \times 3} \end{pmatrix}_{6 \times 6} \quad (17)$$

where $\mathbf{0}_{3 \times 3}$ is a zero matrix of order three and $I_{3 \times 3}$ is an identity matrix of order three. The remaining matrices are specified as

$$\Omega = \begin{pmatrix} \Omega_{xx}^{(0)} & \Omega_{xy}^{(0)} & \Omega_{xz}^{(0)} \\ \Omega_{xy}^{(0)} & \Omega_{yy}^{(0)} & \Omega_{yz}^{(0)} \\ \Omega_{xz}^{(0)} & \Omega_{yz}^{(0)} & \Omega_{zz}^{(0)} \end{pmatrix}, \quad \chi = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (18)$$

where $\Omega_{xx}^{(0)}, \Omega_{xy}^{(0)}, \Omega_{xz}^{(0)}, \dots, \Omega_{zz}^{(0)}$ indicate partial derivatives of order two of the potential at the libration point $(x_0, 0, z_0)$. Specifically, they produce

$$\begin{aligned} \Omega_{xy}^{(0)} &= \Omega_{yx}^{(0)} = \Omega_{yz}^{(0)} = \Omega_{zy}^{(0)} = 0, \quad \Omega_{xz}^{(0)} = \Omega_{zx}^{(0)}, \\ \Omega_{xx}^{(0)} &= 1 - k \left[\frac{q_1(1-\mu)}{r_{10}^3} + \frac{\mu}{r_{20}^3} - \frac{3q_1(1-\mu)(x_0 + \mu)^2}{r_{10}^5} - \frac{3\mu(x_0 + \mu - 1)^2}{r_{20}^5} \right], \\ \Omega_{zx}^{(0)} &= 3k \left[\frac{q_1(1-\mu)(x_0 + \mu)}{r_{10}^5} + \frac{\mu(x_0 + \mu - 1)}{r_{20}^5} \right] z_0, \\ \Omega_{yy}^{(0)} &= 1 - k \left[\frac{q_1(1-\mu)}{r_{10}^3} + \frac{\mu}{r_{20}^3} \right], \\ \Omega_{zz}^{(0)} &= -k \left[\frac{q_1(1-\mu)}{r_{10}^3} + \frac{\mu}{r_{20}^3} - 3 \left(\frac{q_1(1-\mu)}{r_{10}^5} + \frac{\mu}{r_{20}^5} \right) z_0^2 \right], \end{aligned} \quad (19)$$

The characteristic equation of matrix B is as in Vincent *et al.*, (2024a)

$$\lambda^6 + p\lambda^4 + q\lambda^2 + r = 0 \quad (20)$$

where

$$\begin{aligned} p &= 4 - \Omega_{xx}^{(0)} - \Omega_{yy}^{(0)} - \Omega_{zz}^{(0)}, \\ q &= \Omega_{xx}^{(0)}\Omega_{yy}^{(0)} + \Omega_{yy}^{(0)}\Omega_{zz}^{(0)} + \Omega_{zz}^{(0)}\Omega_{xx}^{(0)} - 4\Omega_{zz}^{(0)} - (\Omega_{xz}^{(0)})^2, \\ r &= (\Omega_{xz}^{(0)})^2\Omega_{yy}^{(0)} - \Omega_{xx}^{(0)}\Omega_{yy}^{(0)}\Omega_{zz}^{(0)}. \end{aligned} \quad (21)$$

The characteristic equation (20) is a polynomial equation in λ of degree six and its roots decide the stability behavior of libration points. A libration point is stable in the sense of Lyapunov if all the roots of (20) are pure imaginary numbers or all the roots have negative real parts, else, it is not stable.

On determining the eigen values of the characteristic equation (20), it is established that in the range of permissible values of μ , a couple of roots is entirely imaginary numbers, and the remaining two couples of roots are conjugate complex numbers, having the imaginary portion near to ± 1 for sufficiently small values of k and q_1 . Note that the libration points $L_{1,2}^z$ have at least a complex eigen value with non-negative real part, it is concluded that the libration points $L_{1,2}^z$ are unstable. This result concurs with Ragos and Zagouras, (1988); Ragos and Zagouras, (1993); Xuetaang and Lizhong, (1993); Kalantonis *et al.*, (2021) and Das *et al.*, (2009) in the case of zero force provided by the rod.

This fact appears in Tables 4 and 5 for values of $\mu = 0.1$, $\mu = 0.24$ and a broad scope of values of q_1 and k . Only the two values of μ are shown here, for other values of μ give similar behavior. Since both L_1^z and L_2^z have complex roots with positive real parts (stability type: center $\times$ complex saddle), they are unstable.

Table 4. The coordinates of the libration points $L_{1,2}^z$ and the corresponding eigenvalues $\lambda_{1,2}, \lambda_{3,4,5,6}$ as functions of q_1 and k for fixed value of the mass parameter, $\mu = 0.1$

k	q_1	$L_{1,2}^z (x_0, \pm z_0)$	$\lambda_{1,2}$	$\lambda_{3,4,5,6}$
0.1	-0.07	$(-0.001894, \pm 1.48949)$	$\pm 0.038837i$	$\pm (0.027306 \pm 0.999996i)$
1	-0.05	$(-0.032916, \pm 1.10783)$	$\pm 0.209442i$	$\pm (0.136947 \pm 0.998409i)$
3	-0.03	$(-0.124175, \pm 0.86693)$	$\pm 0.513035i$	$\pm (0.306627 \pm 0.981029i)$
6	-0.01	$(-0.273931, \pm 0.55540)$	$\pm 0.779810i$	$\pm (0.530239 \pm 0.988485i)$

Table 5. The coordinates of the libration points $L_{1,2}^z$ and the corresponding eigenvalues $\lambda_{1,2}, \lambda_{3,4,5,6}$ as functions of q_1 and k for fixed value of the mass parameter, $\mu = 0.24$

k	q_1	$L_{1,2}^z(x_0, \pm z_0)$	$\lambda_{1,2}$	$\lambda_{3,4,5,6}$
0.1	-0.07	$(-0.027735, \pm 0.53623)$	$\pm 0.224904 i$	$\pm (0.143092 \pm 0.997589 i)$
1	-0.05	$(-0.175585, \pm 0.59692)$	$\pm 0.734918 i$	$\pm (0.397586 \pm 0.942350 i)$
3	-0.03	$(-0.363925, \pm 0.55925)$	$\pm 0.935098 i$	$\pm (0.619044 \pm 0.972631 i)$
6	-0.01	$(-0.572346, \pm 0.27332)$	$\pm 1.015450 i$	$\pm (0.558480 \pm 0.892376 i)$

Even though unstable, it is worth noticing that the imaginary portion of the ‘Krein quartet’ of roots of these libration points as the rotating mass dipole tends to zero, is ± 1 for sufficiently small k . This signifies that the motion of the third body m round the libration points is near 1:1 resonance with the turning of the primaries. This effect is depicted in Fig. 8 where we demonstrate that for a scope of values z in-between $z=0.560413$ (correlated to $k=100$, slow rotating mass dipole) and $z=0.534001$ (corresponding to $k=0.1$, fast rotating mass dipole), the imaginary parts of the ‘Krein quartet’ of eigenvalues stay close to ± 1 or fall within the unit disk for $q_1 = -0.0001$ and $\mu = 9.5 \times 10^{-4}$.

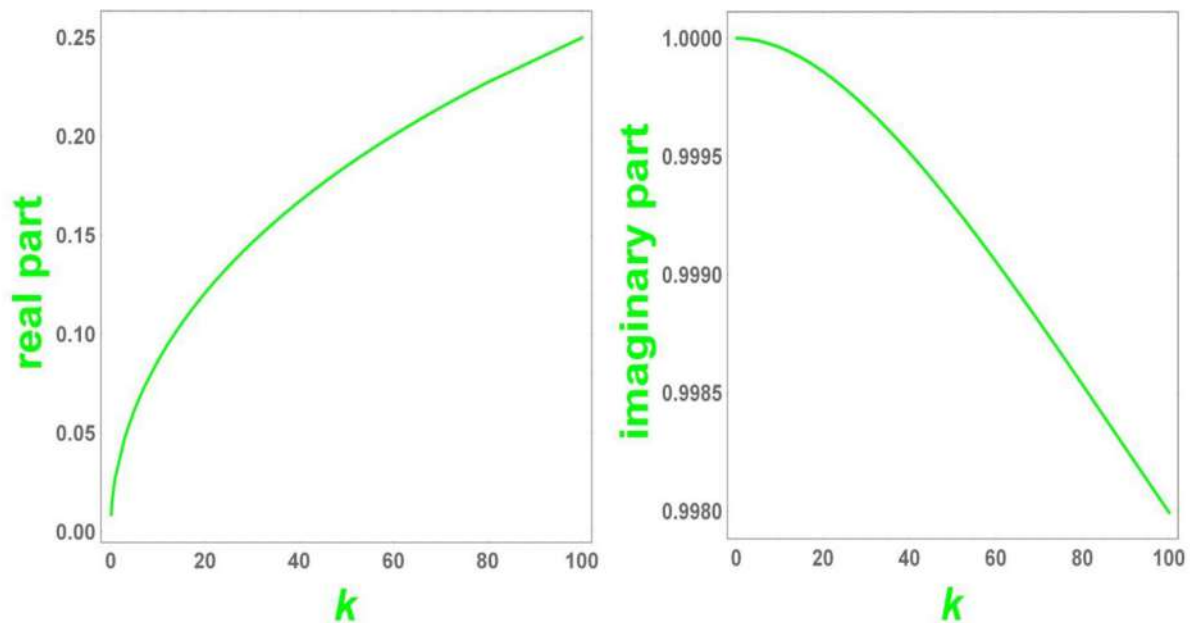


Figure 8. The reliance of the imaginary part (right) and real part (left) of the Krein quartet of roots for the libration points $L_{1,2}^z$ when $q_1 = -0.0001$, $\mu = 9.5 \times 10^{-4}$ and varying $k \in [0.1, 100]$. The perpendicular axis represents the absolute value of the imaginary portion (right) and the real portion (left) of the complex root.

CONCLUSION

The movement of a third body of infinitesimal mass m under the attraction of gravity impact of primary bodies in the structure of the circular restricted three-body problem (rotating mass dipole configuration) when the larger primary radiates while the smaller one does not was studied. The work focused on the presence, location and linear stability of libration points on

the out-of-plane constrained by the influence of the force ratio k , radiation factor q_1 , and mass parameter μ . It is found that the space-based format of the equations of movement allows the existence of out-of-plane libration points if the radiation factor q_1 is non-positive. There exists two out-of-plane libration points positioned in the $(x-z)$ plane in harmonious locations pertaining to the $(x-y)$ plane. The existence and location of these points are influenced by the μ , q_1 and k parameters. The findings show that μ , q_1 and k parameters bring the location of the out-of-plane libration points closer to the line joining the primaries (i.e., the Ox axis).

Furthermore, the regions of possible motion for the particle under the perturbations of photo gravitational and force ratio were discussed. It is established that neighboring the more massive primary body m_1 and through the libration points, the zero-velocity curves mold little ovals of areas not permitted to movement and diminish as the parameter q_1 raises from -1 up to 0 . In contrast, as the rotating mass dipole increases, all the areas are permitted for movement apart from the section in-between the out-of-plane libration points which expand due to the high values of the force ratio k . Thus, the perturbing forces significantly influence the topology of the ZVCs in the Oxz -plane.

Finally, stability analysis in linear sense of these libration points was discussed. A numerical computation of the roots of characteristic equation of the system across a wide range of radiation, force ratio and mass parameters values revealed that, in general, the libration points have at least a complex conjugates roots with positive real part; and hence it is concluded that the libration points $L_{1,2}^z$ are unstable

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REFERENCES

- Agbogidi, B. R., Vincent, E. A., Afere, B. A. E. and Ibrahim, S. D. (2025). Equilibrium points in the perturbed rotating mass dipole system with oblateness, *New Astronomy*, 116, 102344, <https://doi.org/10.1016/j.newast.2024.102344>
- Chermnykh, S. V. (1987). Stability of libration points in a gravitational field, *Leningr. Univ. Vestn. Mat. Mekhanika Astron.*, 2, 73–77
- Das, M. K., Narang, P., Mahajan, S. and Yuasa, M. (2009). On out of plane equilibrium points in photo- gravitational restricted three-body problem, *Journal of Astrophysics and Astronomy*, 30, 177-185. <https://doi.org/10.1007/s12036-009-0009-6>.
- Godziewski, K. and Maciejewski, A. J. (1999). Unrestricted planar problem of a symmetric body and a point mass: triangular libration points and their stability, *Celest. Mech. Dyn. Astron.*, 75, 251–285 <https://doi.org/10.1023/A:1008337017789>
- Gousidou-Koutita, M., and Kalvouridis, T. J. (2009). On the efficiency of Newton and Broyden numerical methods in the investigation of the regular polygon problem of $(N + 1)$ bodies, *Appl. Math. Comput.*, 212, 100112. <https://doi.org/10.1016/j.amc.2009.02.015>
- Gozdziewski, K. (2003). Stability of the triangular libration points in the unrestricted planar problem of a symmetric rigid body and a point mass, *Celest. Mech. Dyn. Astron.*, 85, 79-103. <https://doi.org/10.1023/A:1021742409252>
- Hirabayashi, H., Morimoto, Y. M. Yano, H. Kawaguchi, J. and Bellerose, J. (2010). Linear stability of collinear equilibrium points around an asteroid as a two-connected-mass:

- application to fast rotating asteroid 2000EB14, *Icarus* 206, 780-782. doi:10.1016/j.icarus.2009.12.023
- Idrisi, M. J. and Ullah, M. S. (2024). Exploring out-of-plane equilibrium points in CRTBP: Theoretical insights and empirical observations, *Chaos Solitons and Fractals*, 185 115180-115188. <https://doi.org/10.1016/j.chaos.2024.115180>.
- Jat, R. M. and Kishor, R. (2025). Impact of dark matter halo and albedo effect on the out-of-plane motion in the generalized photogravitational RTBP, *Physics of the Dark Universe*, 48, <https://doi.org/10.1016/j.dark.2025.101940>
- Joshi, M. and Hasan, S. N. (2023). Existence and regions of location of out of plane equilibrium points for photo-gravitational circular restricted three-body problem, in: American Institute of Physics Conference Series, Vol. 2649 of American Institute of Physics Conference Series, AIP, p.030034. <https://doi.org/10.1063/5.0119273>.
- Kalantonis, V. S., Vincent, A. E. Gyegwe, J. M and Perdios, E. A. (2021). Periodic Solutions Around the Out-Of-Plane Equilibrium Points in the Restricted Three-Body Problem with Radiation and Angular Velocity Variation in Th. M. Rassias and P. M. Pardalos, *Springer Optimization and Its Applications*, 167, 251275, <https://doi.org/10.1007/978-3-030-61732-5-11>
- Kalvouridis, T., Arribas, M. and Elife, A. (2006). Dynamical properties of the restricted four-body problem with radiation pressure, *Mech. Res. Commun.*, 33, 811-817. DOI:10.1016/j.mechrescom.2006.01.008
- Leke, O., Cyril-Okeme, V., Stephen, S. and Gyegwe, J. (2025). Investigation of motion around out-of-plane points in the restricted three-body problem with variable shape and masses, *New Astronomy*, 114. 102311. <https://doi.org/10.1016/j.newast.2024.102311>.
- Moneer, E. M., Allawi, Y. M. Alanazi, M. M. and Zotos, E. E. (2023). Revealing the properties of the out-of-plane points of equilibrium of the restricted 3-body problem with non-spherical radiating bodies, *New Astronomy*, 103 102061. <https://doi.org/10.1016/j.newast.2023.102061>.
- Murray, C. D. and Dermott, S. F. (1999). *Solar System Dynamics*, Cambridge University Press, Cambridge, 63-129
- Papadakis, K. E. (2005). Numerical Exploration of Chermnykhs Problem, *Astrophys. Space Sci.*, 299, 6781. <https://doi.org/10.1007/s10509-005-3070-x>
- Papadouris, J. P. and Papadakis, K. E. (2013). Equilibrium points in the photogravitational restricted four-body problem, *Astrophys. Space Sci.*, 344, 2138. DOI 10.1007/s10509-012-1319-8
- Perdios, E. A., Kalantonis, V. S., Perdiou, A. E. and Nikaki, A. A. (2015). Equilibrium Points and Related Periodic Motions in the Restricted Three-Body Problem with Angular Velocity and Radiation Effects, *Adv. Astron.*, 473483. <http://dx.doi.org/10.1155/2015/473483>
- Perdios, E. A. Kalantonis, V. S. and Douskos, C. N. (2008). Straight-line oscillations generating three-dimensional motions in the photogravitational restricted three-body problem, *Astrophys. Space Sci.*, 314, 199208 DOI:10.1007/s10509-008-9755-1
- Perezhogin, A. A. (1976). Stability of the sixth and seventh libration points in the photogravitational restricted circular three-body problem, *Sov. Astron. Lett.*, 2, 174175
- Prieto-Llanos, T. and Gmez-Tierno, M. A. (1994). Station keeping at libration points of natural elongated bodies, *J. Guid. Control Dyn.* 17, 787-794 <https://doi.org/10.2514/3.21268>
- Radzievskii, V. (1950): The restricted problem of three bodies taking account of light pressure, *Astron.*, 27, 250256).
- Radzievskii, V. V. (1953). The space photogravitational restricted three-body problem, *Astron. Z.*, 30, 265-273.
- Ragos, O. and Zagouras, C. G. (1988). Periodic solutions about the out of plane equilibrium points in the photogravitational restricted three-body problem, *Celest. Mech.*, 44, 135-

- 154 .<https://doi.org/10.1007/BF01230711>.
- Ragos, O. and Zagouras, C. G. (1993). On the existence of the out of plane equilibrium points in the photogravitational restricted three-body problem, *Astrophys. Space Sci.*, 209, 267-271 . <https://doi.org/10.1007/BF00627446>.
- Schuerman, D. (1980). The restricted three-body problem including radiation pressure, *Astrophys. J.*, 238, 337342. <https://doi.org/10.1086/157989>
- Simmons, J., McDonald, A. and Brown, J. (1985). The restricted 3-body problem with radiation pressure, *Celest. Mech.*, 35, 145187.<https://doi.org/10.1007/BF01227667>
- Singh, J. an. Vincent, E. A. (2015). Out-of-plane equilibrium points in the photogravitational restricted four-body problem, *Astrophysics and Space Science*, 359, 38 . DOI 10.1007/s10509-015-2487-0
- Vincent, E. A., Abouelmagd, E. I., Perdios, A. E. and Kalantonis, V. S. (2024a). Numerical exploration of the quantized Hill problem dynamics, *Chaos, Solitons and Fractals*, 181, 114688 <https://doi.org/10.1016/j.chaos.2024.114688>
- Vincent, E. A., Perdiou, A. E. and Singh, (2024b). Out-of-Plane Equilibrium Points in the Restricted Three-Body Problem with Radiation Pressure, PoyntingRobertson Drag and Angular Velocity Variation. In T M Rassias and P Pardalos (Eds), *Mathematical Analysis, Differential Equations and Applications*, 903933 .https://doi.org/10.1142/9789811267048_0030
- Vincent, E. A., Tsirogiannis, G. A. Perdiou , A. E and Kalantonis, V. S. (2024c). Equilibrium points and Lyapunov families in the circular restricted three-body problem with an oblate primary and a synchronous rotating dipole secondary: Application to Luhman-16 binary system. *New Astronomy*, *New Astronomy* 105, 102108 .<https://doi.org/10.1016/j.newast.2023.102108>
- Wang, X. Y., Jiang, Y. and Gong, S. P. (2014). Analysis of the potential field and equilibrium points of irregular-shaped minor celestial bodies, *Astrophys. Space. Sci.*, 353, 105121.DOI 10.1007/s10509-014- 2022-8
- Xuetang, X. and Lizhong, Y. (1993). Photogravitational restricted three-body problem and coplanar libration points, *Chinese physical letters*, 10, 61-64 <https://doi.org/10.1088/0256-307X/10/1/017>
- Zeng, X., Baoyin, H. and Li, J. (2016a). Updated rotating mass dipole with oblateness of one primary (II): out-of-plane equilibria and their stability, *Astrophys. Space Sci.*, 15, 361 DOI 10.1007/s10509-015-2599-6
- Zeng, X., Fang, B., Li, J. and Yu, Y. (2016b). Generalized flyby trajectories around elongated minor celestial bodies as a rotating mass dipole, *Acta Mechanica Sinica*, 32, 535-545, DOI10.1007/s10409-015-0487-5
- Zeng, X., Gong, S., Li, J. and Alfriend, K. (2016c). Equilibrium points of elongated celestial bodies as the perturbed rotating mass dipole, *J. Guid. Control Dyn.* 39, 1223
- Zeng, X., Jiang, F. H., Li, J. F. and Baoyin, H. X. (2015). Study on the connection between the rotating mass dipole and natural elongated bodies, *Astrophys. Space Sci.* 356, 29-42.DOI 10.1007/s10509- 014-2187-1