

Article

Out-of-Plane Equilibrium Points in the Photogravitational Hill Three-Body Problem

Aguda E. Vincent ¹, George A. Tsirogiannis ², Angela E. Perdiou ³ and Vassilis S. Kalantonis ^{4,*}

¹ Department of Mathematics, School of Basic Sciences, Nigeria Maritime University Okerenkoko, Warri 331005, Delta State, Nigeria; vincentekele@yahoo.com

² Department of Food Science & Technology, University of Patras, GR-26504 Patras, Greece; gtsirogianni@upatras.gr

³ Department of Civil Engineering, University of Patras, GR-26504 Patras, Greece; aperdiou@upatras.gr

⁴ Department of Electrical & Computer Engineering, University of Patras, GR-26504 Patras, Greece

* Correspondence: kalantonis@upatras.gr

Abstract: This paper investigates the movement of a negligible mass body (third body) in the vicinity of the out-of-plane equilibrium points of the Hill three-body problem under the effect of radiation pressure of the primaries. We study the effect of the radiation parameters through the factors $q_i, i = 1, 2$ on the existence, position, zero-velocity curves and stability of the out-of-plane equilibrium points. These equilibrium positions are derived analytically under the action of radiation pressure exerted by the radiating primary bodies. We determined that these points emerge in symmetrical pairs, and based on the values of the radiation parameters, there may be two along the Oz axis and either none or two on the Oxz plane (outside the axes). A thorough numerical investigation found that both radiation factors have a strong influence on the position of the out-of-plane equilibrium points. Our results also reveal that the parameters have impact on the geometry of the zero-velocity curves. Furthermore, the stability of these points is examined in the linear sense. To do so, the spatial distribution of the eigenvalues on the complex plane of the linearized system is visualized for a wide range of radiation parameter combinations. By a numerical investigation, it is found that all equilibrium points are unstable in general.

Keywords: hill problem; restricted three-body problem; radiating primaries; out-of-plane equilibrium points; zero-velocity curves; linear stability



Citation: Vincent, A.E.; Tsirogiannis, G.A.; Perdiou, A.E.; Kalantonis, V.S. Out-of-Plane Equilibrium Points in the Photogravitational Hill Three-Body Problem. *Symmetry* **2024**, *16*, 1374. <https://doi.org/10.3390/sym16101374>

Received: 3 September 2024

Revised: 4 October 2024

Accepted: 11 October 2024

Published: 15 October 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

One of the most commonly studied problems in Celestial Mechanics is Hill's three-body problem (HTBP), which is a special case of the widespread restricted three-body problem (RTBP). The circular RTBP focuses on the motion of a tiny mass body (third body) within a system consisting of two larger masses (the primaries), which orbit in circular paths around their shared center of mass. The mass of the third body is insignificant in comparison to that of the primary bodies, so its impact on their motion is negligible. On the other hand, the HTBP, originally formulated by Hill [1], aimed to investigate lunar dynamics. In this simplified model, the massless body is also subject to gravitational forces from two primary bodies, but one vastly dominates the other in mass, which is exemplified by the Sun and Earth system. In a rotating reference frame, the smaller primary is fixed at the origin, while the Ox-axis extends toward the larger primary, which is treated as being infinitely distant from the secondary (see, [2,3]). One of the key benefits of the HTBP is that it possesses a significantly simpler Hamiltonian than that of the RTBP, which allows for obtaining easier analytical solutions in certain cases [4–7] while it also provides significant benefits for numerical computations [8–15].

The dynamics of a negligible mass body in orbit around a single star and planet or around a binary star system can be viewed as a generalization of the classical RTBP.

This model is called the photogravitational restricted three-body problem [16–18] and has attracted much interest (see, e.g., [19–24]). When the radiation pressure exerted by the primaries (one or both) is also considered, apart from the five coplanar equilibrium points, additional stationary points positioned outside the physical plane (the plane which is defined by the two primary bodies) have been identified, and the dynamics of orbits around these new points have been discussed in the literature (see, e.g., [25–29]). Several perturbed versions of the RTBP have also been found to possess additional equilibrium points. Many works have been devoted to these modified models and presented both numerical and analytical investigations on the existence and location of the equilibrium points along with their stability towards understanding the dynamical behavior of orbits around them (see, e.g., [30–33]).

On the other hand, Markellos et al. [34] introduced the model of the photogravitational HTBP and offered some initial insights into the problem's formulation, the position of the equilibrium points and zero-velocity curves with estimates of the size of accretion disks in binary stars. The derived model builds upon the classical Hill lunar problem and represents one of its possible extensions. Other models for the HTBP that involve lack of sphericity of body shape (oblate primary) and radiation pressure effects were studied by [35–43], among others. Furthermore, Yárnoz et al. [44] studied the evolution of families a and g of periodic orbits in the HTBP with radiation pressure and found that the main structure of family characteristics differs from the corresponding solution of the classical HTBP.

In particular, Kanavos et al. [35] analyzed the equilibrium points and associated periodic solutions in the photogravitational HTBP, concentrating on the 2D context. In this paper, we extend this analysis to the spatial case of this specific modification of the HTBP by investigating the out-of-plane equilibrium points (OoPEPs) and, in particular, the motion around them. Our objective is to determine the positions of the OoPEPs with respect to the radiation coefficients, the regions of motion as defined by the zero-velocity curves, and the linear stability of these points. We present the spatial distribution of the eigenvalues of the linearized system, showing how they scatter across a systematic grid of radiation factor values. This provides both a qualitative and quantitative overview of how the radiation from the two primaries influences the stability of the OoPEPs. To achieve this, we employ a new two-step technique. In the first step, we discretize the continuous space of the linearized system around the equilibrium points over a dense grid of radiation parameter combinations and numerically compute their eigenvalues. In the second step, we discretize the continuous space of the eigenvalue complex plane and compute the local spatial density. The combination of these two steps yields valuable information regarding the basins of attraction at which the eigenvalues are accumulated.

At this point it should be emphasized that from the above literature and also to the authors' knowledge, no research has been performed on the existence and location of equilibrium points outside the orbital plane defined by the primaries and especially the motion around them on the HTBP with radiating primaries, hence, it sparked our interest in examining the impact of radiation pressure on the dynamical characteristics of these points. We identify two symmetrical pairs of equilibrium points situated along the z axis and Oxz plane, correspondingly. The presence of equilibrium points located outside the plane of the primaries, known as OoPEPs, is a new aspect of the model that had not been addressed in [35]. To tackle the current problem more effectively, we employ both an analytical approach and a numerical treatment.

Our work is organized as follows: In Section 2, we recall the equations of motion of the photogravitational HTBP along with its Jacobian integral. In Section 3, we investigate the OoPEPs of the system with variations in the radiation factors. In Sections 4 and 5, the configurations of the zero velocity curves and linear stability of the OoPEPs are investigated, respectively, while the conclusion and discussion are presented in Section 6. The model under consideration can be utilized to study the motion of minor objects like cosmic dust and grains.

2. Photogravitational Hill's Equations of Motion

In the present analysis, we adopt the terminology and notations associated with the photogravitational HTBP, which were discussed and analyzed in [34,35]. Therefore, the equations governing the motion of the massless body within the context of the spatial photogravitational HTBP are as follows:

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega}{\partial x} = \Omega_x, \ddot{y} + 2\dot{x} = \frac{\partial \Omega}{\partial y} = \Omega_y, \ddot{z} = \frac{\partial \Omega}{\partial z} = \Omega_z, \quad (1)$$

where the photogravitational potential is defined as

$$\Omega = \Omega(x, y, z) = \frac{3x^2}{2} - Q_1x + \frac{q_2}{r} - \frac{z^2}{2}, \quad (2)$$

and

$$r^2 = x^2 + y^2 + z^2, Q_1 = \frac{1 - q_1}{\mu^{\frac{1}{3}}}$$

where r represents the distance between the particle and the smaller primary body. The Jacobian integral associated with this problem, is expressed by the equation

$$\Gamma = -\dot{x}^2 - \dot{y}^2 - \dot{z}^2 + 3x^2 - 2Q_1x + \frac{2q_2}{r} - z^2 \quad (3)$$

where Γ is the Jacobian constant. In this context, μ represents the fraction of the smaller primary's mass relative to the total mass of the primary bodies and $0 < \mu < \frac{1}{2}$. Also, q_1, q_2 are the radiation factors of the primary and secondary body, respectively. Let us recall that according to Schuerman [17], the radiation force effect q can be represented by the relation $q = 1 - b$, where $b = \frac{F_r}{F_g}$ signifies the ratio of the force F_r generated by the radiation to the gravitational force F_g resulting from the two primary bodies. In case where $q = 1$, namely $b = 0$, it reduces to the classical problem that takes into account only gravitational forces while if $0 < q < 1$, that is $b \in (0, 1)$, gravitational force surpasses the radiation pressure. Furthermore, when $q = 0$, namely $b = 1$, the forces of radiation and gravity are in equilibrium and finally, when $q < 0$, i.e., $b > 1$, radiation pressure overrides the gravitational attraction. Consequently, it results that the allowable range of values for the radiation factor in the present problem could be taken $q_{1,2} \in (-\infty, 1]$. It is worth noting that in case of $Q_1 = 0, q_2 = 1$, the problem becomes the classical HTBP (see, e.g., [45]).

3. Existence and Location of the Equilibrium Points of the System

For the existence of any equilibrium point, the necessary and sufficient conditions that must be fulfilled are $\dot{x} = \dot{y} = \dot{z} = \ddot{x} = \ddot{y} = \ddot{z} = 0$, i.e., the velocity and acceleration components must both be zero. As such, the coordinates (x, y, z) that define the equilibrium points can be established through the solution of the equations

$$\Omega_x = 0, \Omega_y = 0, \Omega_z = 0, \quad (4)$$

where $\Omega_x, \Omega_y, \Omega_z$ are the first-order partial derivatives of $\Omega(x, y, z)$ with respect to x, y , and z , respectively. In particular, we have

$$\begin{aligned} \Omega_x &= 3x - Q_1 - \frac{q_2x}{r^3}, \\ \Omega_y &= -\frac{q_2y}{r^3}, \\ \Omega_z &= -z \left[1 + \frac{q_2}{r^3} \right]. \end{aligned} \quad (5)$$

Since in this paper, we are mainly interested in the OoPEPs, we focus on the third equation of System (4). So, this system could potentially possess stationary points outside

the orbital plane when solved for $x \neq 0, y = 0$ and $z \neq 0$. Based in this premise, the third Equation (4) is satisfied if

$$1 + \frac{q_2}{r^3} = 0. \quad (6)$$

By substituting Equation (6) into second of Equations (4), it follows that $y = 0$, that is to say, the equilibrium solution will be on the Oxz plane.

Equation (6) can be expressed as

$$r = (-q_2)^{\frac{1}{3}} \equiv k_0 \quad (7)$$

with $r^2 = x^2 + z^2$. For $q_2 < 0$, we get from Equation (7) that $r > 0$ and the locus of these points describes an Apollonius circle surrounding the body. According to Equation (7), the existence of any real solution of the form $(x, 0, z)$ requires that at least one of the following conditions is met (some values in this region might not yield OoPEPs):

$$q_2 < 0 \text{ or } q_2 = 0. \quad (8)$$

The second condition will not be considered in this paper in order System (4) to have real solutions of the form $(x, 0, z)$ as for $y = 0$ its second equation must be fulfilled. As a result, the equilibrium points outside the orbital plane are positioned on the Oxz plane, necessitating the solution of the other two equations of System (4) for $(x, z) \neq (0, 0)$. That is, the OoPEPs result from the solution of the system

$$\begin{aligned} 3x_0 - Q_1 - \frac{q_2 x_0}{r_0^3} &= 0, \\ 1 + \frac{q_2}{r_0^3} &= 0 \end{aligned} \quad (9)$$

with

$$r_0^2 = x_0^2 + z_0^2. \quad (10)$$

These solutions depend on the radiation factors Q_1 and q_2 , which in turn depend on the problem considered. From the second equation of Equation (9) combined with the first equation, we have

$$x_0 = \frac{Q_1}{4}. \quad (11)$$

Substituting Equation (11) in Equation (10) in combination with Equation (7), we have

$$z_0 = \pm \sqrt{(-q_2)^{\frac{2}{3}} - \frac{Q_1^2}{16}}, \quad \left((-q_2)^{\frac{2}{3}} \geq \frac{Q_1^2}{16} \right). \quad (12)$$

This proves the existence of OoPEPs in pair, namely L_1^z for $z > 0$ and its symmetric point L_2^z for $z < 0$ with respect to the axis Ox . We point out here the symmetry property $z \rightarrow -z$ in Equation (9). As such, if the point L_1^z is an equilibrium point, also L_2^z is. We remark that the points $L_{1,2}^z$ and $L_{3,4}^z$ do not have any real existence on the xz plane simultaneously. So, Equations (11) and (12) are the locations of the OoPEPs of the HTBP under radiating primaries and they solely depend on the radiation factors.

Notably, conditions (9) result in two distinct types of equilibrium points within the Oxz plane. Particularly, one set of these points positioned along the Oz axis ($x_0 = 0$) of the synodic coordinate system symmetrically located with respect to the origin and the physical plane (Oxy), which continue to exist for all values of the coefficients $q_2 < 0$ and $Q_1 = 0$, in contrast to the equilibrium points located on the Oxz plane outside the Ox and Oz axis ($0 < x_0 < |z_0|$ or $x_0 > |z_0|$). The latter stationary points can exist only in the region $\{(q_2, Q_1) \mid q_2 < 0, Q_1 > 0\}$, nonetheless, this is a necessary condition, meaning that there may be values within this region where these points fail to exist. All these pairs of points where exist have the same Jacobian constant.

From Equation (9), it could be obtained that the existence condition of $L_{1,2}^z$ on the Oxz plane is that the radiation factors of the two primaries must satisfy

$$q_2 < \Phi(Q_1) < 0 \text{ where } \Phi(Q_1) = -\frac{Q_1^3}{64}, Q_1 > 0. \quad (13)$$

It must be pointed out that the equilibrium points on the axis Oz and on the Oxz plane will not exist together at the same time (simultaneously) in the photogravitational HTBP, which validates findings of [37] in the photogravitational HTBP with oblateness.

To support the conclusion drawn above, Figure 1 illustrates the planar curves defined by the implicit equations $\Omega_x = 0$ (green) and $\Omega_z = 0$ (red) curves. The points where these curves intersect reveal the (x_0, z_0) coordinates of the equilibrium points within the xz -plane, for two different cases of Q_1 , (a) $Q_1 = 0$ and (b) $Q_1 = 0.5$. The radiation factor q_2 of the smaller primary in both examples have the same value $q_2 = -2$. As previously noted, in the left frame of this figure the equilibrium points are situated along the Oz axis, whereas in frame (b), they are found on the Oxz plane. In both cases, their positions are denoted by blue dots. These numerical results support our conclusion drawn above.

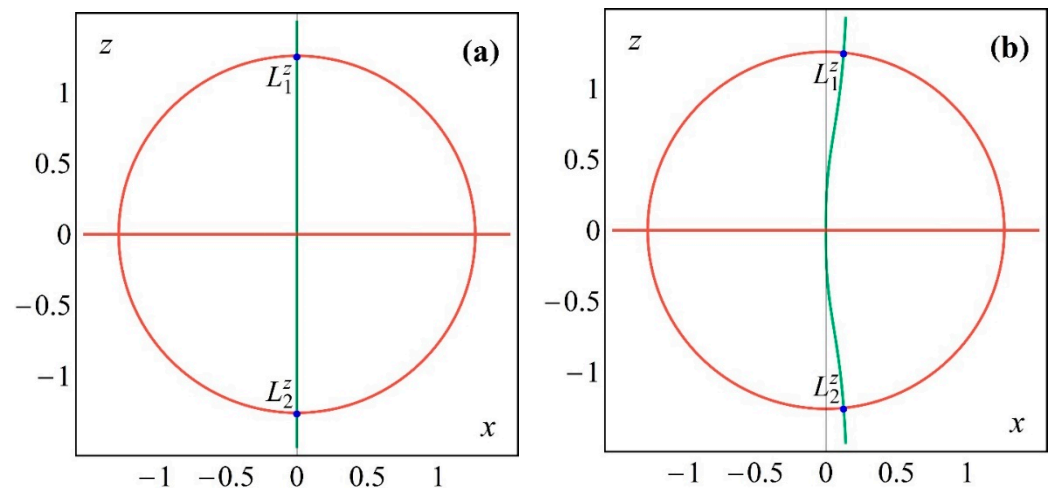
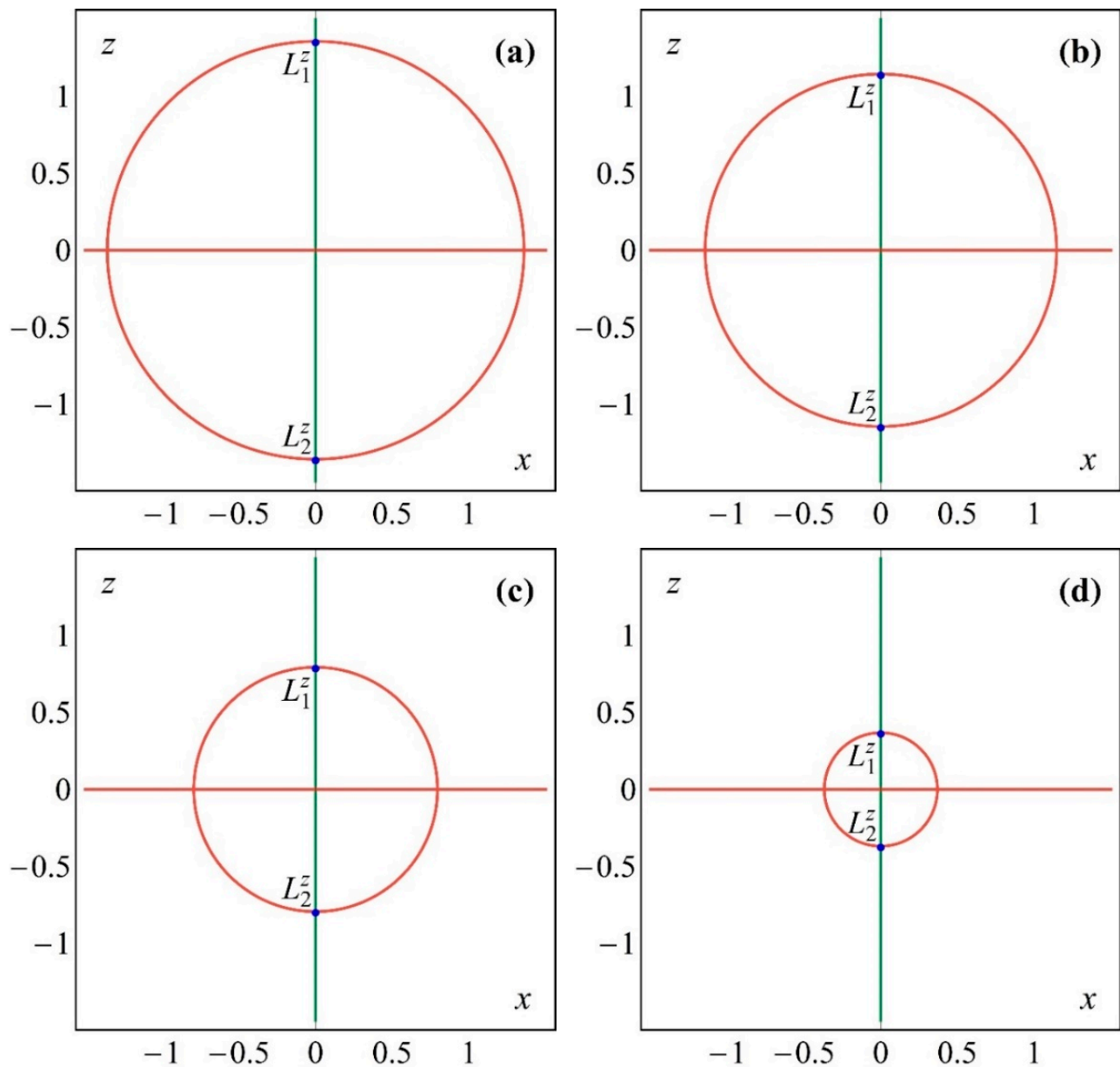


Figure 1. The positions of the two pairs of symmetric points L_1^z and L_2^z on each of the axis Oz and Oxz plane, respectively, as determined by the intersections of equations $\Omega_x = 0$ (green curve) and $\Omega_z = 0$ (red curve), for $q_2 = -2$ and the values $Q_1 = 0$ for (a) and $Q_1 = 0.5$ for (b). Blue dots indicate the positions of the equilibrium points.

Next, we will examine the locations of the OoPEPs for the infinitesimal mass body when one or both primaries are radiating. The respective coordinates are displayed in the second column of Table 1 for a case of only the smaller primary body q_2 radiating when the value $Q_1 = 0$. In this case, the problem admits two out-of-plane critical points on the axis Oz along with the variation q_2 . It is noted that an increase in the radiation factor q_2 leads to a decrease in the z -coordinates of the equilibrium points towards the origin. This effect is shown in Figure 2. It is worth noting that we choose relatively large values of radiation factor q_2 so it can be seen clearly in the figures. In the third and fourth columns of this table we present how the positions of the equilibrium points vary with changes in q_2 but for $Q_1 = 0.35$ and $Q_1 = 0.5$, respectively. In both cases where the primaries radiating, the problem admits two OoPEPs on the xz -plane. As the radiation factor q_2 increases, for fixed $Q_1 = 0.35$, a decrease in the z -coordinates of the equilibrium points toward the origin is noted, while the x -coordinates stay unchanged (maintain their positions). Therefore, it is apparent that the variation in the corresponding positions for $Q_1 = 0.5$ is comparable to the situation previously discussed for $Q_1 = 0.35$.

Table 1. The locations $(x_0, \pm z_0)$ of the OoPEPs L_1^z and L_2^z of the problem when both primaries radiate.

q_2	$Q_1 = 0$	$Q_1 = 0.35$	$Q_1 = 0.5$
−0.05	$(0, \pm 0.36840315)$	$(0.08750000, \pm 0.35786119)$	$(0.12500000, \pm 0.34654853)$
−0.04	$(0, \pm 0.34199519)$	$(0.08750000, \pm 0.33061225)$	$(0.12500000, \pm 0.31833270)$
−0.03	$(0, \pm 0.31072325)$	$(0.08750000, \pm 0.29814877)$	$(0.12500000, \pm 0.28447133)$
−0.02	$(0, \pm 0.27144176)$	$(0.08750000, \pm 0.25695210)$	$(0.12500000, \pm 0.24094736)$
−0.01	$(0, \pm 0.21544347)$	$(0.08750000, \pm 0.19687468)$	$(0.12500000, \pm 0.17547333)$

**Figure 2.** The locations of the symmetric equilibrium points L_1^z and L_2^z on axis Oz as determined by the intersections of equations $\Omega_x = 0$ (green curve) and $\Omega_z = 0$ (red curve), for $Q_1 = 0$ and the values $q_2 = -2.5$ for (a), $q_2 = -1.5$ for (b), $q_2 = -0.5$ for (c) and $q_2 = -0.05$ for (d). Blue dots indicate the position of the equilibrium points.

We observe a similar phenomenon (rows 2 through 6) in Table 1 for fixed values of q_2 and varying Q_1 . The positions of the OoPEPs on the xz -plane are presented in Figure 3, where we show their evolution for $Q_1 = 0.5$ while varying q_2 since the behavior remains

consistent for different values of these parameters. Another difference, with respect to the case of only the smaller primary radiating is that the OoPEPs, L_1^z and L_2^z are consistently found to be located in the second and fourth quadrants, respectively, for every value of q_2 . It is also remarkable that both the existence and number of these points are dependent on the radiation parameters q_1 and q_2 . In the case where $q_2 < 0$ and $Q_1 \neq 0$ ($q_1 \neq 1$), our numerical computations show that we may have zero or two such points on the xz -plane in contrast to the photogravitational restricted three-body problem case (see, e.g., [16,18,28,40]), where either one or two pairs of equilibrium points may be present in the Oxz plane in case where the parameters satisfy the conditions $q_2 > 0$ and $q_1 < 0$ or $q_2 < 0$ and $q_1 > 0$, correspondingly.

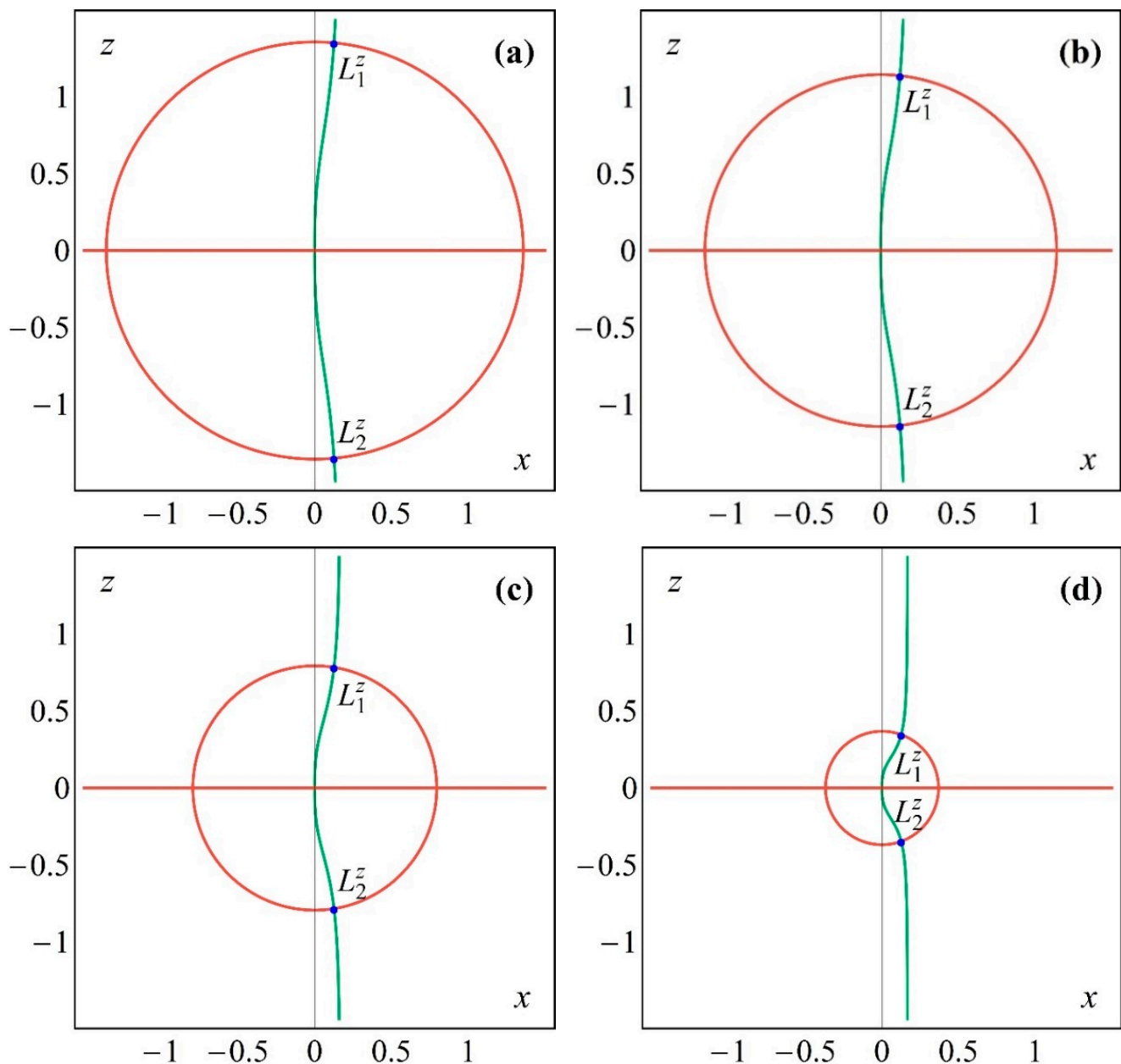


Figure 3. The positions of the symmetric points L_1^z and L_2^z on Oxz plane as determined by the intersections of equations $\Omega_x = 0$ (green curve) and $\Omega_z = 0$ (red curve), for $Q_1 = 0.5$ and the values $q_2 = -2.5$ for (a), $q_2 = -1.5$ for (b), $q_2 = -0.5$ for (c) and $q_2 = -0.05$ for (d). Blue dots indicate the position of the equilibrium points.

Towards an attempt to investigate and visualize, with a systematic way, the effect of Q_1 and q_2 on the positions of OoPEPs, we created the surfaces of Figure 4. In these plots, we evaluated Equations (11) and (12) with a dense two-dimensional grid of a wide range of the radiation parameters, $Q_1 \in (0, 1]$ and $q_2 \in [-5.5, 0]$. The top subfigures show the effect on the position in a decouple way, while the bottom subfigure is a four-dimensional plot. Here, we used a color-code to depict the effect of q_2 on the equilibrium point position. It is clearly observed that lower values of q_2 displace the z direction of the equilibrium points, and Q_1 has more effect on the x direction, while their combination is a nearly plane surface.

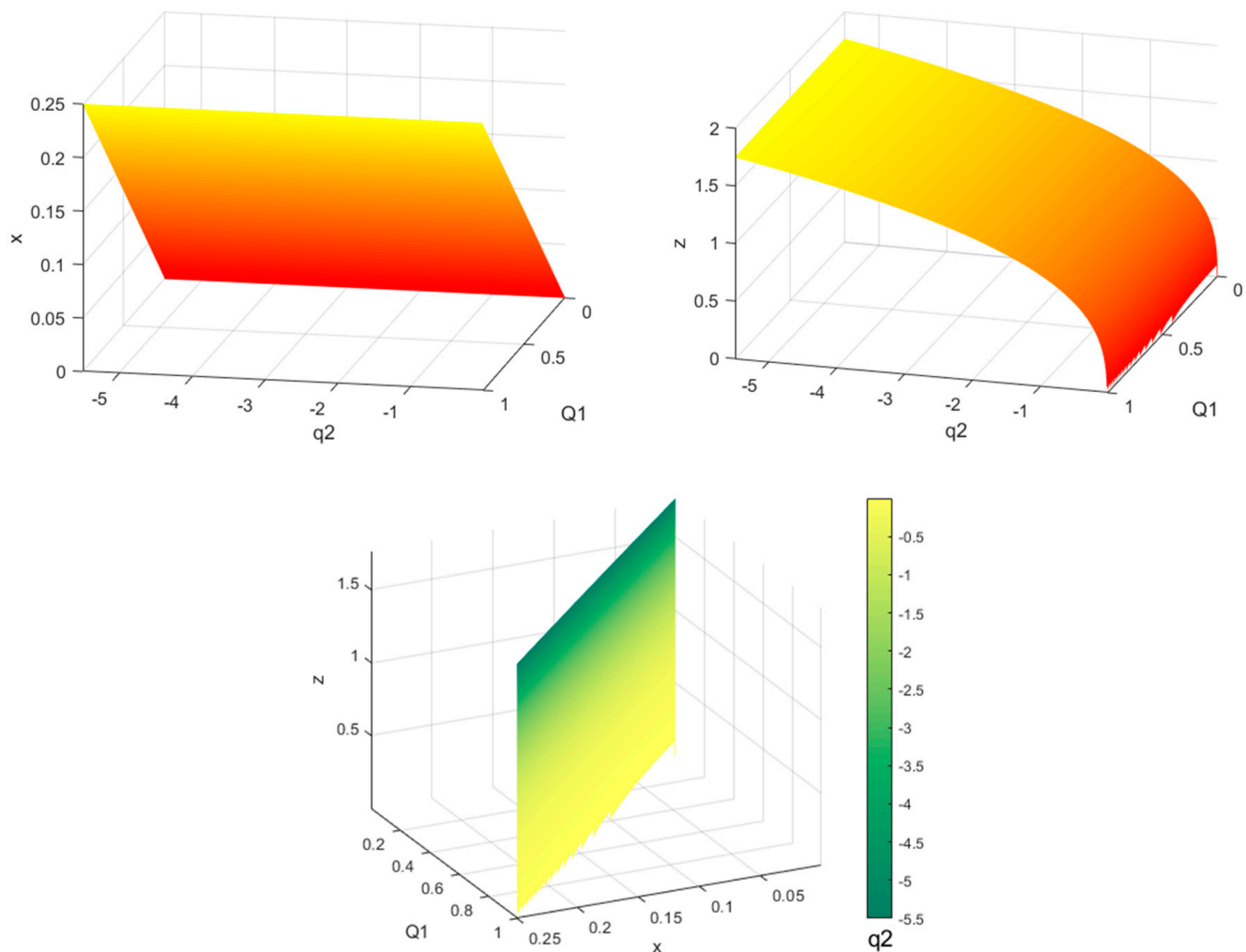


Figure 4. The positions of the symmetric OoPEPs L_1^z and L_2^z on Oxz plane as functions of $Q_1 \in (0, 1]$ and $q_2 \in [-5.5, 0]$.

4. Zero-Velocity Curves in the (x, z) Plane

In this section, we present the contours of the surface (3) on the (x, z) plane for zero velocity, which define the zero velocity curves (ZVCs). In this surface projection, trapping regions—where the motion of the massless body is constrained—form within specific ranges of the Jacobian constant. As the value of Γ varies, these regions can either expand or contract until they eventually reduce to individual points. Figure 5 shows the ZVCs for a fixed value of $Q_1 = 0.5$ and for three different values of the radiation factor, namely, $q_2 = -2.5$, $q_2 = -1$ and $q_2 = -0.05$, respectively.

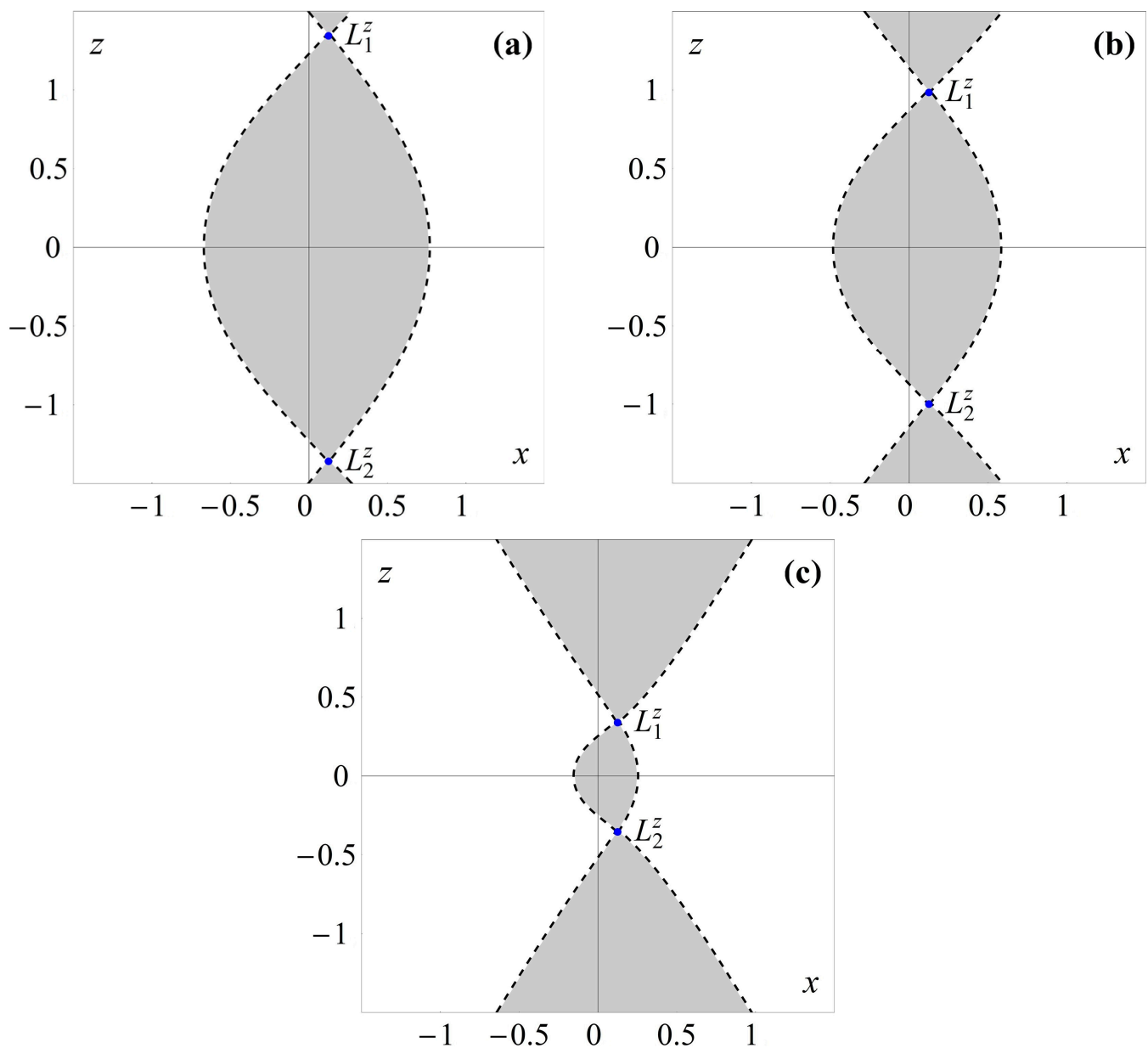


Figure 5. Zero velocity curves in the (x, z) plane and the locations of the OoPEPs are shown for $Q_1 = 0.5$ and (a) $q_2 = -2.5$, (b) $q_2 = -1.0$, and (c) $q_2 = -0.05$, respectively. The gray area represents regions where motion is not possible.

We have plotted only the curves (dotted lines) for Jacobian constant values corresponding to the OoPEP L_1^z (or L_2^z). Blue dots mark the positions of the OoPEPs in the system. In this case, between the OoPEPs, the zero velocity curves form small oval regions where motion is not allowed. These regions shrink as the radiation factor q_2 approaches zero (i.e., as q_2 increases), while the regions of allowed motion expand with the increase in q_2 . This behavior is similar to the restricted three-body problem with radiating primaries and angular velocity variation described in [28]. It is clear from this figure that the radiation factors significantly influence the regions of possible and forbidden motion for the third body.

5. Stability of the Out-of-Plane Equilibrium Points

To examine the linear stability of the OoPEPs L_1^z (or L_2^z), we position the origin at the equilibrium point by setting:

$$x = x_0 + \xi, y = \eta, z = z_0 + \zeta \quad (14)$$

let ξ , η , and ζ represent the perturbations along the Ox , Oy and Oz , axis, respectively. By substituting Equation (14) into System (1) and performing a Taylor series expansion around the equilibrium point, keeping only the first-order terms, we derive the linearized equations of motion as follows:

$$\dot{\chi} = A\chi, \chi = (\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}, \dot{\zeta})^T. \quad (15)$$

Let χ represent the state vector of the third particle relative to the equilibrium points, and the time-independent coefficient matrix A is given by

$$A = \begin{pmatrix} 0_{3 \times 3} & I_{3 \times 3} \\ \Omega_{3 \times 3} & B_{3 \times 3} \end{pmatrix}_{6 \times 6}. \quad (16)$$

Here, $0_{3 \times 3}$ and $I_{3 \times 3}$ represent the third-order zero matrix and the identity matrix. The two additional matrices are defined as follows:

$$\Omega = \begin{pmatrix} \Omega_{xx}^{(0)} & \Omega_{xy}^{(0)} & \Omega_{xz}^{(0)} \\ \Omega_{yx}^{(0)} & \Omega_{yy}^{(0)} & \Omega_{yz}^{(0)} \\ \Omega_{zx}^{(0)} & \Omega_{zy}^{(0)} & \Omega_{zz}^{(0)} \end{pmatrix}, B = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (17)$$

where $\Omega_{xx}^{(0)}, \Omega_{xy}^{(0)}, \Omega_{xz}^{(0)}, \dots, \Omega_{zz}^{(0)}$ denote the second partial derivative of the system potential of the OoPEP (x_0, z_0) .

The characteristic equation of system described in Equation (15) is thus written as

$$\lambda^6 + a_1\lambda^4 + a_2\lambda^2 + a_3 = 0 \quad (18)$$

where

$$\begin{aligned} a_1 &= 4 - \Omega_{xx}^{(0)} - \Omega_{yy}^{(0)} - \Omega_{zz}^{(0)}, \\ a_2 &= \Omega_{xx}^{(0)}\Omega_{yy}^{(0)} + \Omega_{yy}^{(0)}\Omega_{zz}^{(0)} + \Omega_{zz}^{(0)}\Omega_{xx}^{(0)} - 4\Omega_{zz}^{(0)} - (\Omega_{xz}^{(0)})^2, \\ a_3 &= (\Omega_{xz}^{(0)})^2\Omega_{yy}^{(0)} - \Omega_{xx}^{(0)}\Omega_{yy}^{(0)}\Omega_{zz}^{(0)}. \end{aligned} \quad (19)$$

By calculating the second-order derivatives of Ω at the equilibrium point, we obtain the following:

$$\begin{aligned} \Omega_{xy}^{(0)} &= \Omega_{yx}^{(0)} = \Omega_{yz}^{(0)} = \Omega_{zy}^{(0)} = 0, \Omega_{xz}^{(0)} = \Omega_{zx}^{(0)}, \\ \Omega_{xx}^{(0)} &= 3 - \frac{q_2}{r_0^3} + \frac{3q_2x_0^2}{r_0^5}, \\ \Omega_{zz}^{(0)} &= -1 - \frac{q_2}{r_0^3} + \frac{3q_2z_0^2}{r_0^5}, \\ \Omega_{zx}^{(0)} &= \frac{3q_2x_0z_0}{r_0^5}, \\ \Omega_{yy}^{(0)} &= -\frac{q_2}{r_0^3}. \end{aligned} \quad (20)$$

Equation (18) is a sixth-degree polynomial in λ , and the resulting eigenvalues indicate the linear stability or instability of the corresponding OoPEPs. An equilibrium point is considered linearly stable if all six roots of the characteristic equation are purely imaginary or have non-positive real parts; otherwise, the point is deemed unstable.

We analyze the nature of the roots of Equation (18) and find that for all admissible values Q_1, q_2 , one pair of eigenvalues is purely imaginary, while the other two pairs are complex conjugates with positive real parts. The presence of complex roots with positive

real parts indicates that motion around the OoPEPs is unstable, similar to cases previously examined within the framework of the restricted three-body problem (see, e.g., [28,37,46] and references therein).

To clearly illustrate this fact, we compute the stability of the equilibrium points along the Oz axis and those on the Oxz plane, as shown in Tables 2 and 3, respectively, over a wide range of radiation parameters. Our analysis reveals that there is no instance where all the eigenvalues are purely imaginary. Therefore, we conclude that the positive real parts of the complex roots contribute to the instability of the OoPEPs.

Table 2. The stability of $L_{1,2}^z$ on the Oz axis as a function of q_2 for $Q_1 = 0$.

q_2	$L_{1,2}^z(x_0, \pm z_0)$	$\lambda_{1,2}$	$\lambda_{3,4,5,6}$
−5.5	(0, ± 1.765174168)	$\pm 1.732050807i$	$\pm(1.118033988 \pm 0.8660254037i)$
−4.0	(0, ± 1.587401052)	$\pm 1.732050808i$	$\pm(1.118033989 \pm 0.8660254038i)$
−2.0	(0, ± 1.259921050)	$\pm 1.732050807i$	$\pm(1.118033989 \pm 0.8660254038i)$
−1.0	(0, ± 1.000000000)	$\pm 1.732050808i$	$\pm(1.118033989 \pm 0.8660254038i)$
−0.01	(0, ± 0.215443469)	$\pm 1.732050808i$	$\pm(1.118033989 \pm 0.8660254038i)$
−0.005	(0, ± 0.170997595)	$\pm 1.732050807i$	$\pm(1.118033988 \pm 0.8660254037i)$

Table 3. The stability of $L_{1,2}^z$ on the Oxz plane as a function of q_2 for $Q_1 = 0.5$.

q_2	$L_{1,2}^z(x_0, \pm z_0)$	$\lambda_{1,2}$	$\lambda_{3,4,5,6}$
−5.5	(0.12500000, ± 1.760742696)	$\pm 1.730962960i$	$\pm(1.116769707 \pm 0.8654814802i)$
−4.0	(0.12500000, ± 1.582471832)	$\pm 1.730705035i$	$\pm(1.116469855 \pm 0.8653525173i)$
−2.0	(0.12500000, ± 1.253704930)	$\pm 1.729911465i$	$\pm(1.115547067 \pm 0.8649557326i)$
−1.0	(0.12500000, ± 0.992156742)	$\pm 1.728647057i$	$\pm(1.114076068 \pm 0.8643235285i)$
−0.01	(0.12500000, ± 0.175473327)	$\pm 1.647106082i$	$\pm(1.017211303 \pm 0.8235530413i)$
−0.005	(0.12500000, ± 0.116684092)	$\pm 1.579922392i$	$\pm(0.933871550 \pm 0.7899611956i)$

A numerical study of Equation (16), conducted on a dense grid with parameters $Q_1 \in (0, 1]$ and $q_2 \in [-5.5, 0)$, provides insights into the stability of the OoPEPs. We numerically evaluate matrix A and compute its eigenvalues. The distribution of these eigenvalues for all parameter combinations is presented in Figure 6. This plot reveals a broad range of corresponding eigenvalues; while some individual eigenvalues fall within the unit disk, none of the parameter combinations result in stable equilibrium points (i.e., at least one pair of eigenvalues indicates instability).

Although this figure provides an overview of the range of the eigenvalues, it lacks information regarding their density on the complex plane, an important piece of information about the role of the radiation parameters. An estimation of the density is computed by using a honeycomb-like pattern of regular hexagons as shown in Figure 7 (left subplot). This way, we obtain a discretization of the complex plane that acts as a reference for the estimation of the spatial density of the eigenvalues. Each computed eigenvalue of A for every parameter combination is assigned to a single hexagon. By counting the total number of eigenvalues that belong to each hexagon with respect to the total number of eigenvalues, we estimate the local density. In the right subplot of Figure 7, we observe a clear high density on six locations (bright yellow hexagons), i.e., nearly all matrices have those six particular eigenvalues except a very small fraction.

This observation calls for a further investigation. Here, we try to answer the following question: Which combinations of Q_1 and q_2 cause this deviation? In Figure 8, with red color, we show the combinations of the radiation parameters that lead to at least one-pair eigenvalue deviation. We observe the following patterns: (a) values of $Q_1 \approx 1$ in combination with $q_2 \approx 0$ and (b) values of $Q_1 \approx 0$ lead to different eigenvalues than all other combinations.

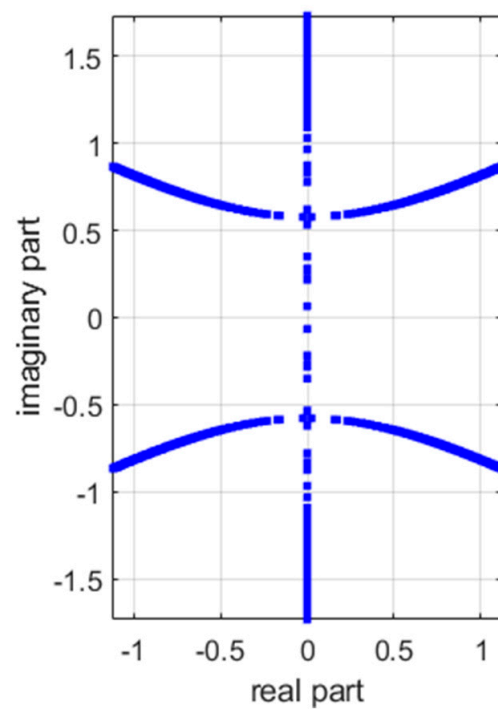


Figure 6. Location of the eigenvalues in the complex plane of matrix A (Equation (16)) for a dense grid for parameters $Q_1 \in (0, 1]$, $q_2 \in [-5.5, 0)$.

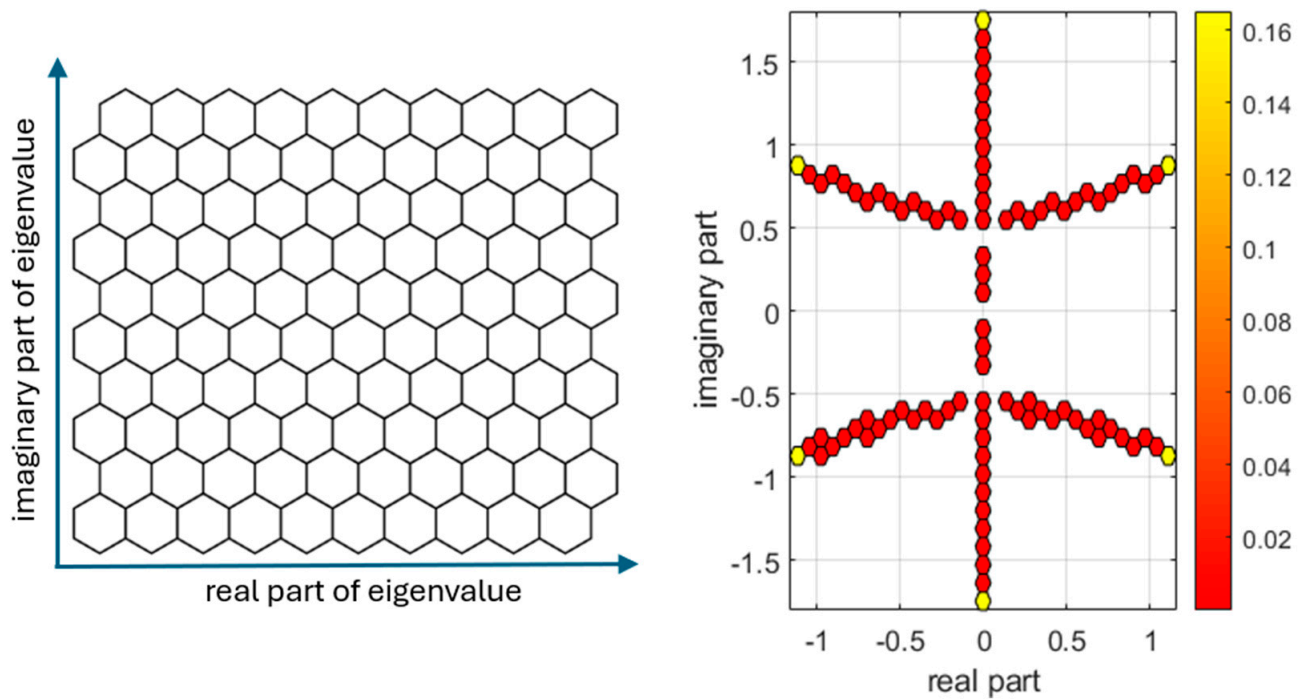


Figure 7. Discretization pattern of the complex plane and spatial density distribution for parameters $Q_1 \in (0, 1]$, $q_2 \in [-5.5, 0)$.

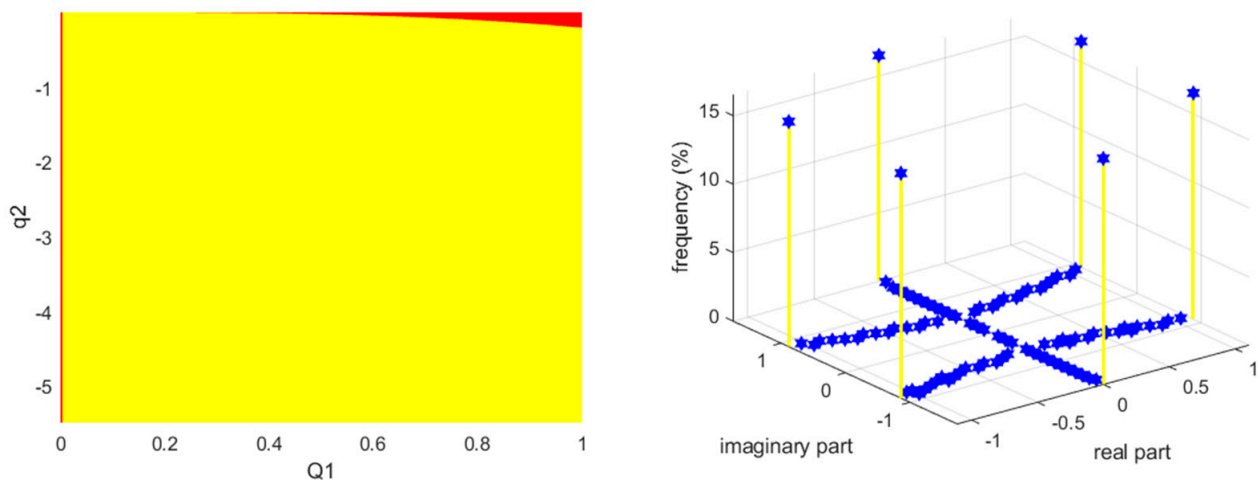


Figure 8. Combinations of the radiation parameters (left) and the spatial density of locations (right) of the eigenvalues in the complex plane of matrix A (Equation (16)) (dense grid for parameters $Q_1 \in (0, 1]$, $q_2 \in [-5.5, 0)$, red color indicates divergence of the eigenvalues).

6. Discussion and Conclusions

We investigated the Hill three-body problem (HTBP) with radiating primaries, utilizing the equations of motion established by Kanavos et al. [35]. Our focus was on the existence and location of the out-of-plane equilibrium points (OoPEPs), revealing that the spatial form of the dynamical equations admits this kind of point when the radiation factor q_2 of the secondary body is negative. We examined two scenarios: first, when the smaller primary radiates while the larger primary does not, and second, when both primaries are radiating. Notably, these points do not have corresponding ones in the classical HTBP.

Initially, we analyzed a pair of equilibrium points positioned along the z -axis when the larger primary does not radiate. In the case when both primaries radiate, we found two such points in the Oxz plane. The radiation factors significantly influence the existence of equilibrium points, leading to situations where either zero or two OoPEPs exist in symmetrical positions relative to the (x, y) plane. The locations of these points are affected by the radiation factors that express the perturbing forces. Our findings indicate that as the radiation factor of the smaller primary approaches zero (from negative values) while the radiation factor of the larger primary remains fixed, the OoPEPs converge toward the location of the secondary, which is the origin.

We examined the regions of motion for the particle as defined by the zero-velocity surface, yielding corresponding equipotential curves. Our analysis revealed various closed or trapped regions where the infinitesimal mass can move freely, depending on the values of the Jacobian constant Γ . It was noted that the regions permitting motion for the infinitesimal particle expand as the radiation factor q_2 increases. The trapped motion regions (non-permissible for motion) diminish in size as the radiation factor q_2 increases for fixed values of the radiation factor Q_1 . Thus, radiation pressure forces significantly impact the shape of the zero-velocity curves in the (x, z) plane.

We also assessed the linear stability of these equilibrium points across a broad range of radiation parameter values and determined that the motion of the infinitesimal mass around the OoPEPs is unstable. This finding reinforces the conclusions of [28,29,37,46], and others regarding the circular restricted three-body problem under the influence of radiation pressure from the primaries. A thorough numerical analysis of matrix A (Equation (16)) revealed that, in general, the eigenvalues are concentrated around six dominant complex values, while a narrow area of radiation parameter value combinations results in divergent eigenvalues.

Finally, we note that the simpler form of the Hamiltonian for the HTBP enables us to analytically study the positions of the stationary points as well as their stability. In contrast,

such analyses in the restricted three-body problem are feasible only through numerical methods (see, e.g., [18,28,46]). A similar investigation into the photogravitational HTBP with oblateness has been conducted by Markakis et al. [37]. In this context, the oblateness of the secondary body allows for the existence of OoPEPs, which are situated solely in the $(x - z)$ plane with $x \neq 0$.

Author Contributions: A.E.V.: Conceptualization, Methodology, Validation, Visualization, Formal analysis, Writing—original draft, Writing—review & editing. G.A.T.: Conceptualization, Methodology, Validation, Visualization, Formal analysis, Writing—original draft, Writing—review & editing. A.E.P.: Conceptualization, Methodology, Validation, Visualization, Formal analysis, Writing—original draft, Writing—review & editing. V.S.K.: Conceptualization, Methodology, Validation, Visualization, Formal analysis, Writing—original draft, Writing—review & editing. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: All data are within the paper.

Acknowledgments: The authors would like to thank the three unknown reviewers for their constructive comments and suggestions that significantly improved this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

- Hill, G.W. Researches in the lunar theory. *Am. J. Math.* **1878**, *1*, 5–26. [\[CrossRef\]](#)
- Tsirogiannis, G.A.; Perdios, E.A.; Markellos, V.V. Improved grid search method: An efficient tool for global computation of periodic orbits: Application to Hill's problem. *Celest. Mech. Dyn. Astron.* **2009**, *103*, 49–78.
- Liu, K.Y.-Y.; Villac, B. Periodic orbits families in the Hill's three-body problem with solar radiation pressure. *Adv. Astronaut. Sci.* **2010**, *136*, 285–300.
- de Bustos, M.T.; Lopez, M.A.; Martinez, R.; Vera, J.A. On the periodic solutions emerging from the equilibria of the Hill Lunar problem with oblateness. *Qual. Theory Dyn. Syst.* **2018**, *17*, 331–344.
- Aydin, C. The Conley–Zehnder indices of the spatial Hill three-body problem. *Celest. Mech. Dyn. Astron.* **2023**, *135*, 32.
- Frauenfelder, U.; Moreno, A. On GIT quotients of the symplectic group, stability and bifurcations of periodic orbits (with a view towards practical applications). *J. Symplect. Geom.* **2023**, *21*, 723–773. [\[CrossRef\]](#)
- Xu, X. Determination of the doubly symmetric periodic orbits in the restricted three-body problem and Hill's lunar problem. *Celest. Mech. Dyn. Astron.* **2023**, *135*, 8.
- Hénon, M. Numerical exploration of the restricted problem V. Hill's case: Periodic orbits and their stability. *Astron. Astrophys.* **1969**, *1*, 223–238.
- Zagouras, C.; Markellos, V.V. Three-dimensional periodic solutions around equilibrium points in Hill's problem. *Celest. Mech.* **1985**, *35*, 257–267.
- Villac, B.F.; Scheeres, D.J. Escaping trajectories in the Hill three-body problem and applications. *J. Guid. Control Dyn.* **2003**, *26*, 224–232.
- Zotos, E.E. Comparing the fractal basins of attraction in the Hill problem with oblateness and radiation. *Astrophys. Space Sci.* **2017**, *362*, 190. [\[CrossRef\]](#)
- Kalantonis, V.S. Numerical Investigation for Periodic Orbits in the Hill Three-Body Problem. *Universe* **2020**, *6*, 72. [\[CrossRef\]](#)
- Asano, Y.; Satoh, S.; Yamada, K. Analysis of period-multiplying bifurcations of distant retrograde orbits in the Hill three-body problem. *Adv. Space Res.* **2022**, *70*, 3016–3033. [\[CrossRef\]](#)
- Moneer, E.M.; Alanazi, M.; Elaissi, S.; Allawi, Y.; Dubeibe, F.L.; Zotos, E.Z. Orbital dynamics in the Hill problem with oblateness. *Results Phys.* **2023**, *53*, 106936. [\[CrossRef\]](#)
- Ibrahim, M.K.; Rabeh, T.; Abouelmagd, E.I. Dynamical Properties of Perturbed Hill's System. *Math. Comput. Appl.* **2024**, *29*, 66. [\[CrossRef\]](#)
- Radzievskii, V. The space photogravitational restricted three-body problem. *Astron. Zh* **1953**, *30*, 265–269.
- Schuerman, D. The restricted three-body problem including radiation pressure. *Astrophys. J.* **1980**, *238*, 337–342. [\[CrossRef\]](#)
- Simmons, J.; McDonald, A.; Brown, J. The restricted 3-body problem with radiation pressure. *Celest. Mech.* **1985**, *35*, 145–187. [\[CrossRef\]](#)
- Suraj, M.S.; Aggrawal, R.; Shalini, K.; Asique, M.C. Out-of-plane equilibrium points and regions of motion in photogravitational R3BP when the primaries are heterogeneous spheroids with three layers. *New Astron.* **2018**, *63*, 15–26. [\[CrossRef\]](#)
- Suraj, M.S.; Aggrawal, R.; Asique, M.C.; Mittal, A. The effect of radiation pressure on the basins of convergence in the restricted four-body problem. *Chaos Soliton Fract.* **2020**, *141*, 110347. [\[CrossRef\]](#)

21. Wang, R.; Wang, Y.; Gao, F. Bifurcation analysis and approximate analytical periodic solution of ER3BP with radiation and albedo effects. *Astrophys. Space Sci.* **2021**, *366*, 29. [[CrossRef](#)]
22. Moneer, E.M.; Allawi, Y.M.; Alanazi, M.; Zotos, E.Z. Revealing the properties of the out-of-plane points of equilibrium of the restricted 3-body problem with non-spherical radiating bodies. *New Astron.* **2023**, *103*, 102061. [[CrossRef](#)]
23. Sheth, D.; Pathak, N.M.; Thomas, V.O.; Abouelmagd, E.I. Periodic orbits analysis of elliptical Sun-Saturn system. *Astron. Rep.* **2023**, *67*, 520–535. [[CrossRef](#)]
24. Gahlot, P.; Kishor, R. Invariant manifolds of Lyapunov periodic orbits in the RCD solar sail problem with dipole secondary. *Nonlinear Dyn.* **2024**, *112*, 14143–14157. [[CrossRef](#)]
25. Ragos, O.; Zagouras, C.G. Periodic solutions about the “out of plane” equilibrium points in the photogravitational restricted three-body problem. *Celest. Mech.* **1988**, *44*, 135–154. [[CrossRef](#)]
26. Ragos, O.; Zagouras, C.G. On the existence of the “out of plane” equilibrium points in the photogravitational restricted three-body problem. *Astrophys. Space Sci.* **1988**, *209*, 267–271. [[CrossRef](#)]
27. Gao, F.; Wang, R. Bifurcation analysis and periodic solutions of the HD 191408 system with triaxial and radiative perturbations. *Universe* **2020**, *6*, 35. [[CrossRef](#)]
28. Kalantonis, V.S.; Vincent, E.A.; Gyegwe, J.M.; Perdios, E.A. Periodic solutions around the out-of-plane equilibrium points in the restricted three-body problem with radiation and angular velocity variation. In *Nonlinear Analysis and Global Optimization*; Rassias, T.M., Pardalos, P.M., Eds.; Springer: Cham, Switzerland, 2021; Volume 167, pp. 251–275.
29. Vincent, E.A.; Singh, J. Out-of-plane equilibria in the perturbed Photogravitational restricted three-body problem with Poynting-Robertson (P-R) drag. *Heliyon* **2022**, *8*, e09603. [[CrossRef](#)]
30. Suraj, M.S.; Aggarwal, R.; Aggarwal, V.K.; Asique, M.C. Combined effect of small perturbations in the Coriolis and centrifugal forces and three-body interaction on the existence of stationary points in the R3BP. *New Astron.* **2021**, *89*, 101630. [[CrossRef](#)]
31. Suraj, M.S.; Aggarwal, R.; Asique, M.C.; Mittal, A. On the modified circular restricted three-body problem with variable mass. *New Astron.* **2021**, *84*, 101510. [[CrossRef](#)]
32. Suraj, M.S.; Aggarwal, R.; Asique, M.C.; Mittal, A. On the Basins of Convergence in the Magnetic-Binary Problem with Angular Velocity. *Comp. Math. Methods* **2021**, *3*, 1161. [[CrossRef](#)]
33. Suraj, M.S.; Aggarwal, R.; Asique, M.C.; Mittal, A. The influence of third order terms on basins of convergence in the Hénon-Heiles type system. *New Astron.* **2022**, *94*, 101761. [[CrossRef](#)]
34. Markellos, V.V.; Roy, A.E.; Velgakis, M.J.; Kanavos, S.S. A photogravitational Hill problem and radiation effects on Hill stability of orbits. *Astrophys. Space Sci.* **2000**, *271*, 293–301. [[CrossRef](#)]
35. Kanavos, S.S.; Markellos, V.V.; Perdios, E.A.; Douskos, C.N. The photogravitational Hill problem: Numerical exploration. *Earth Moon Planets* **2002**, *91*, 223–241. [[CrossRef](#)]
36. Papadakis, K.E. The planar photogravitational Hill problem. *Int. J. Bifurc. Chaos* **2006**, *16*, 1809–1821. [[CrossRef](#)]
37. Markakis, M.P.; Perdiou, A.E.; Douskos, C.N. The photogravitational Hill problem with oblateness: Equilibrium points and Lyapunov families. *Astrophys. Space Sci.* **2008**, *315*, 297–306. [[CrossRef](#)]
38. Douskos, C. Collinear equilibrium points of Hill’s problem with radiation and oblateness and their fractal basins of attraction. *Astrophys. Space Sci.* **2010**, *326*, 263–271. [[CrossRef](#)]
39. Perdiou, A.E.; Perdios, E.A.; Kalantonis, V.S. Periodic orbits of the Hill problem with radiation and oblateness. *Astrophys. Space Sci.* **2012**, *342*, 19–30. [[CrossRef](#)]
40. Kalantonis, V.S.; Perdiou, A.E.; Douskos, C.N. Asymptotic orbits in Hill’s problem when the larger primary is a source of radiation. In *Applications of Nonlinear Analysis*; Rassias, T.M., Ed.; Springer: Cham, Switzerland, 2018; Volume 134, pp. 523–535.
41. Bairwa, L.K.; Pal, A.K.; Kumari, R.; Alhowaity, S.; Abouelmagd, E.I. Study of Lagrange points in the earth–moon system with continuation fractional potential. *Fractal Fract.* **2022**, *6*, 321. [[CrossRef](#)]
42. Abouelmagd, E.I. New model for Hill’s problem in the framework of continuation fractional potential. *Math. Comput. Appl.* **2024**, *29*, 34. [[CrossRef](#)]
43. Vincent, E.A.; Abouelmagd, E.I.; Perdios, A.E.; Kalantonis, V.S. Numerical exploration of the quantized Hill problem dynamics. *Chaos Solitons Fractals* **2024**, *181*, 114688. [[CrossRef](#)]
44. Yárnoz, D.G.; Scheeres, D.J.; McInnes, C.R. On the a and g families of orbits in the Hill problem with solar radiation pressure and their application to asteroid orbiters. *Celest. Mech. Dyn. Astron.* **2015**, *121*, 365–384. [[CrossRef](#)]
45. Szebehely, V. *Theory of Orbits*; Academic Press: New York, NY, USA, 1967.
46. Xuetang, Z.; Lizhong, Y. Photogravitational restricted three-body problem and coplanar libration points. *Chin. Phys. Lett.* **1993**, *10*, 61–64.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.