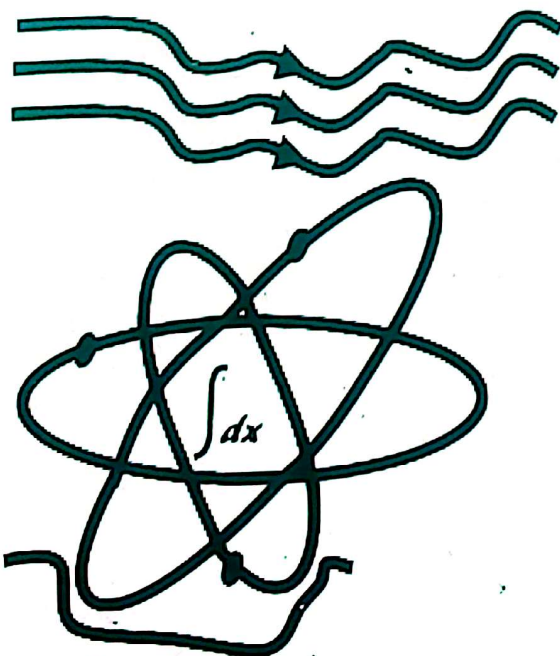


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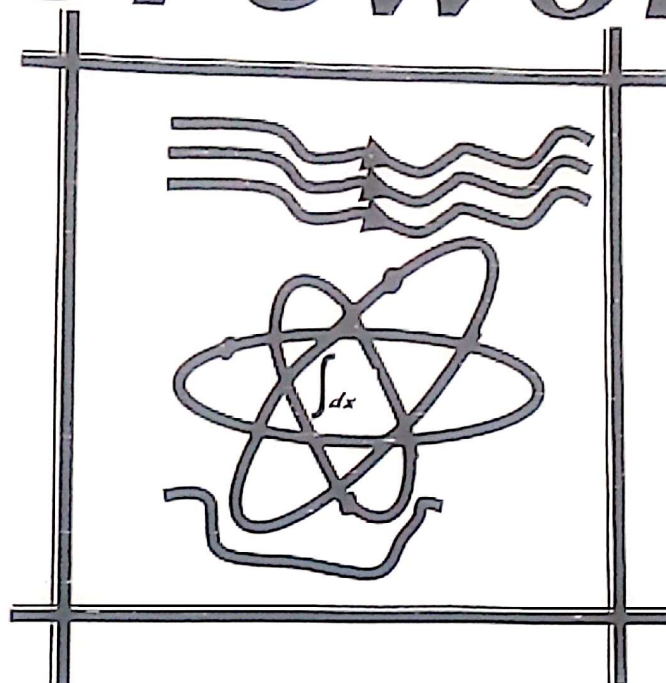
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Foreword



In the foreword to the Journal of NAMP, whose maiden issue came out in 1992, it has been noted that there were only very few scientists in Nigeria who could strictly speaking be called Mathematical Physicists. For this reason, it was decided that the term Mathematical Physics should be broadened to mean: any Scientist who applies fairly rigorous mathematics to Physics, Chemistry, Engineering, or other sciences and also any Mathematician whose results have direct applicability in Physics, Chemistry, Engineering and so forth. In a great country like Nigeria, the process of growth which led to the formation of NAMP in 1979 has produced, as at today, quite a number of scientists whose works bear signatures of the true meaning of the term Mathematical Physicist. To capture such works is the burden of this second Journal of NAMP, called Transactions of the Nigerian Association of Mathematical Physics.

It is my sincere hope that several scientists will try to have their works published in the Transactions of NAMP.

Prof. G. Babaji

January, 2019.

EFFECT OF PERTURBATIONS IN CORIOLIS AND CENTRIFUGAL FORCES ON THE EQUILIBRIUM POINTS IN THE PHOTOGRAVITATIONAL RESTRICTED FOUR-BODY PROBLEM: OUT-OF-PLANE CASE

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Abstract

The photogravitational restricted four-body problem including the effect of perturbations in Coriolis and centrifugal forces is employed to describe the motion of an infinitesimal particle near the out-of-plane equilibrium points in the vicinity of three finite radiating bodies. The three bodies (P_1, P_2, P_3) are moving in circular orbits about their common centre of mass fixed at the origin of the coordinate system, according to the solution of Lagrange where they are always at the vertices of an equilateral triangle. The fourth body P_4 of infinitesimal mass does not affect the motion of the three bodies. We consider that two of the bodies (P_2 and P_3) have the same radiation and mass value μ while the dominant primary body P_1 is of mass $1 - 2\mu$. The equilibrium points (L_1, L_2) lying out of the orbital plane of the three bodies as well as the allowed regions of motion as determined by the zero velocity curves are studied numerically. It is observed that their positions as well as the allowed regions of motion while not affected by the small perturbation in the Coriolis force, are essentially varied under the joint effects of radiation pressure parameters and centrifugal force. Finally the stability of these points is studied and they are found to be unstable.

Keywords: Equilibrium points; Restricted four-body problem; Coriolis and centrifugal forces; Radiation; Zero velocity curves; Stability.

1. Introduction

The Restricted four-body problem (R4BP) is perhaps the simplest model after its famous predecessor, the restricted three-body problem (R3BP) and a natural generalization of it. It deals with the motion of an infinitesimal particle under the Newtonian gravitational attraction of three bodies, called primaries, whose trajectories are the solution of the three Newtonian body problems. It is known that in the planar general problem of three bodies attracting each other according to Newtonian gravitational law, there exist only two permanent central configurations, namely, the collinear (Eulerian) and the equilateral (Lagrangian). In the first case, the primaries of the problem lie on a single straight line while in the second one, the primary bodies lie at the vertices of an equilateral triangle. In analogy with the R3BP, we consider here the restricted problem of four bodies consisting of three primaries moving in circular orbits keeping an equilateral triangle configuration and a massless particle moving under the gravitational attraction of the primaries. The R4BP has witnessed intensive research contributions in the scientific community resulting into several applications such as the Sun-Jupiter and Saturn system [1]; star, two massive planets and a massless Trojan [2]; star, brown dwarf, gas giant and a massless Trojan [3]; Jupiter, Trojan Asteroid, spacecraft [4]; Spacecraft orbit design in the circular restricted three-body problem [5]. Applications of Gravity Assists in the Bicircular and Bielliptic R4BP and references therein. In the same vein, many authors and scientists have extended these ideas in the R4BP: The existence of equilibrium points and their stability was treated by [6, 7]. The radiation effect of the problem was carried out by [8–13]. Also, [14] studied the problem with oblate primaries. Positions of the equilibrium points of the problem with triaxiality of the primaries was studied [15–17] and effect of the Coriolis and centrifugal forces of the problem has been investigated [18, 19].

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$$\Omega_{\text{ext}}^0 = \beta \left[\frac{q_1(1-2\mu)}{r_{10}^3} \right] - \frac{q_1(\epsilon_1 + \sqrt{3}\mu)^2}{r_{10}^3} - \frac{2q_1\mu}{r_{20}^3} - \frac{q_1(\epsilon_2 - \sqrt{3}(1-2\mu))^2}{r_{20}^3}$$

$$\Omega_1 = 3\alpha \left[\frac{q_1(1-2\mu)(\epsilon_1 + \sqrt{3}\mu)}{r_1} + \frac{2\mu_1\mu(\epsilon_1 - \sqrt{3}(1-2\mu))}{r_2} \right]$$

$$\Omega_p^0 = \mu - \frac{q_1(1-2\mu)}{r_{10}} + \frac{2\mu q_2}{r_{20}} \left| \frac{3}{4} - r_{20}^2 \right|$$

$$\Omega^{\pm} = \frac{(1-2\mu)q_1}{r_{10}} \left(\frac{z_0^{\frac{1}{2}}}{r_{10}^{\frac{1}{2}}} - 1 \right) + \frac{2\mu l_2}{r_{20}} \left(\frac{z_0^{\frac{1}{2}}}{r_{20}^{\frac{1}{2}}} - 1 \right)$$

with

$$r_{10}^2 = (x_0 + \sqrt{3}\mu)^2 + z_0^2, \quad r_{20}^2 = r_{30}^2 = (x_0 - \frac{\sqrt{3}}{2}(1-2\mu))^2 + \frac{1}{4} + z_0^2, \quad q_2 = q_1$$

(13)

with

$$a = 4\alpha^2 - \Omega^0_{XX} - \Omega^0_{YY} - \Omega^0_{ZZ}$$

$$b = \Omega^6 x \Omega'' y + \Omega'' y \Omega^6 z z + \Omega^6 z z \Omega^6 x - 4\Omega^6 z z - (\Omega^6 x)^2$$

$$c = (\Omega^0 x_2)^2 \Omega^0 y - \Omega^0 x \Omega^0 y \Omega^0 z z$$

which is a polynomial of sixth degree in λ .

The eigenvalues of the characteristic equation (17) determine the stability or instability of the respective equilibrium point. The characteristic equation has six imaginary roots or complex roots with non-zero

An equilibrium point will be stable if the characteristic equation has six purely real parts.

We have computed the characteristic roots $\lambda_i, i = 1, 2, \dots, 6$ as the perturbation parameter ϵ approaches zero. For a sufficiently small step and found no case in which all the roots are

$\alpha = 1 + \varepsilon_1, |\varepsilon_1| \ll 1$ and $\beta = 1 + \varepsilon_2, |\varepsilon_2| \ll 1$) with an arbitrary

imaginary. This lead to the instability of the out of plane points.

6. **Discussion and conclusion**

In this paper we have studied the existence, location and the stability of the out of plane equilibrium points for particles moving in the vicinity of three massive bodies which emit light radiation under the influence of small perturbations in the Coriolis and centrifugal forces formulated on the basis of Lagrange equilateral triangle configuration. As it is known, the points do not appear if only the gravitational forces are considered. Solutions of equations (8) and (9) give the positions of the out of plane points, they are affected by the radiation pressure and the perturbation in centrifugal force, but not by the Coriolis force because equations (8) and (9) are independent of parameter α . There are two out of plane equilibrium points that lie in the $(x-z)$ plane in symmetrical positions with respect to the (x, y) plane. The effects of the parameters involved on the positions of the out of plane points are shown numerically (Tables 1-3) and graphically in Figures 1-3. The existence of these points is of particular astronomical interest in connection with planetary system formation, satellite motion, etc. The involved parameters is also seen to have significant effects on the topology of the zero velocity curves in the $(x-z)$ plane (Fig. 4 panels (a)-(c)). In particular, between the centre of the dominant primary body and its companion out of plane equilibrium points, the zero velocity curves form small rhombus of regions not allowed to motion which shrink very fast under the variation of radiation pressures and the particle comes very close to P_1 . Finally, the stability of these points has been achieved numerically by determining the roots of the characteristic equation. The numerical investigation of these points shows no case in which the roots are all imaginary in spite of the introduction of aforementioned parameters. Consequently the motion is unbounded, and thus unstable, which agree with [8] when only the dominant primary body is a radiation source and with [10] in the absence of Coriolis and centrifugal forces.

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