



The influence of external magnetic and Aharonov–Bohm flux fields on bound states of the Klein–Gordon and Schrodinger equations via the SWKB approach

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Abstract. We obtained the approximate two-dimensional bound states solutions of a Klein–Gordon particle moving in a quantum mechanical solvable potential under the influence of the external magnetic and Aharonov–Bohm flux fields. We utilized the supersymmetric WKB approach to obtain both the energy levels and normalized wave function in closed form. The proposed potential reduces to the screened Kratzer, Kratzer, Yukawa and Coulomb potential functions as special cases and admits the corresponding energy eigenvalues in both relativistic and non-relativistic regimes. When the external magnetic and Aharonov–Bohm flux fields were turned off, the relativistic, non-relativistic energies and ground state probability density overlap for the magnetic quantum numbers ($m = -1, 1$). The presence of the fields removes the degeneracy and shifts the energy levels of the Klein–Gordon particle. However, we found that the magnetic field strength has no effects on the maximum non-relativistic energy. Generally, the results are consistent with the works in existing literature where the authors utilized different forms of potential energy functions.

1 Introduction

The study of the motion of charged particles under the influence of Aharonov–Bohm (AB) and external magnetic flux fields in two-dimensional space using relativistic and non-relativistic wave equations for considerable potentials has grabbed the interest of researchers for decades [1–3]. The AB effect is a phenomenon in quantum mechanics that is defined by Hamiltonian formalism, as opposed to Newtonian physics, which uses gauge transformations to explain the electromagnetic fields [4]. In 1959, Aharonov and Bohm (AB) published a paper with the title “Significance of Electromagnetic Potentials in the Quantum Theory.” They predicted that electromagnetic potentials ϕ and $\vec{A}$ might be real physical quantities that could be demonstrated in electron interference experiments [5]. The AB effects occur

when a moving charge is changed by scalar and vector potentials which appear in the wave equations. The magnetic field may cause the Zeeman [6] effects or result to energy splitting and shifting. The exact and approximate bound state solutions of the Schrodinger and Klein–Gordon equations under the influence of AB flux and external magnetic fields have been studied by various researchers. Ikot et al. [7] studied superstatistics of Schrodinger equation with pseudo-harmonic potential in external magnetic and AB fields where they utilized the effective Boltzmann’s factor in the superstatistics formalism to obtain partition function and thermodynamic properties such as Helmholtz free energy, internal energy, entropy and specific heat capacity of diatomic molecular systems. Ikot et al. [8] used the factorization approach to investigate the thermodynamic characteristics of AB flux and magnetic fields with screened Kratzer potential in the Schrodinger equation. The system’s eigenvalue and eigenfunction were provided in closed form. With a small screening parameter, their findings suggest that the presence of a magnetic field

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largely reduces the degeneracy. The influence of uniform magnetic and AB flux fields on effective potentials and energy spectra with the SE and KGE under the harmonic oscillator was investigated by Ikhdair et al. [9]. They discovered that the effective potentials and energy levels of the SE and KGE fluctuate in the presence of both magnetic and AB flux fields. Within the framework of the asymptotic iteration approach, Aygun et al. [10] investigated the effect of magnetic field on the energy spectrum of a particle traveling under Kratzer potential. Purohit et al. [11] studied bound state solutions and thermodynamic properties of the screened cosine Kratzer potential under the influence of the magnetic field and AB flux field using the parametric Nikiforov–Uvarov method. They utilized a Greene–Aldrich approximation to deal with the centrifugal barrier term. Furthermore, they presented the eigenenergy equation in a closed form and applied it to obtain the partition function and other thermo-magnetic properties such as magnetization, magnetic susceptibility and persistent current. Eshghi et al. [12] also studied the energy state of the Hulthen plus Coulomb-like potential with position-dependent mass function in an external magnetic field. Edet and Ikot [13] investigated the topological defect on the energy spectra and thermo-magnetic properties as applied to carbon (II) oxide molecule with screened modified Kratzer potential using asymptotic iteration method. They discovered that the confinement effect of AB flux and external magnetic fields on this potential model exhibits both paramagnetic and diamagnetic behavior for weak and intense magnetic fields, respectively. Wang [14] studied the absence of the electric AB effect due to induced charges. In this study, we will investigate the effect of the AB flux and external magnetic field on Klein–Gordon particle using a quantum mechanical solvable potential. We proposed a Varshni-type potential [15] given by

$$U(r) = \left(A - \frac{B}{r}\right) \left(C - \frac{De^{-\alpha r}}{r}\right) \quad (1)$$

where

A, B, C and D are adjustable parameters and α is the potential range. The Varshni potential is useful for the modeling of rotational–vibrational and dissociation energies of diatomic molecules [16]. Also, the Varshni potential has been used to access thermodynamic properties [17], Fishers’ information theoretic measures [18] and the scattering solutions of the Klein–Gordon and Duffin–Kemmer–Petiau equations [19, 20]. In this study, we modeled the potential in such a way that we could obtain the special cases of other potentials by adjusting the parameters instead of solving the individual potentials. The potential reduces to other well-known potential functions by the substitution of the appropriate parameters. Examples include the screened Kratzer potential [21] if $C = 0, D = 1$.

$$U(r) = \left(-\frac{A}{r} + \frac{B}{r^2}\right) e^{-\alpha r} \quad (2)$$

The potential in (2) reduces to the respective Kratzer ($\alpha = 0$) [22], Yukawa ($B = 0$) [23] and the Coulomb’s ($\alpha = B = 0$) [24] potentials given by

$$U(r) = \left(-\frac{A}{r} + \frac{B}{r^2}\right) \quad (3)$$

$$U(r) = -\frac{Ae^{-\alpha r}}{r} \quad (4)$$

$$U(r) = -\frac{A}{r} \quad (5)$$

The Klein–Gordon equation (KGE) for a charged particle moving in a uniform magnetic and AB flux fields is given [25] by

$$\left\{c^2 \left(-i\hbar\nabla + \frac{e}{c}\vec{A}\right)^2 - (E_{nm} - U(r))^2 + (M_0c^2 + S(r))^2\right\} \psi_{nm}(r, \phi) \quad (6)$$

where $c, e, E_{nm}, M_0, U(r)$ and $S(r)$ are the respective speed of light, electron charge, two-dimensional energy spectra, relativistic rest mass, repulsive vector and attractive scalar potentials. The vector potential $\vec{A}$ is given by $\vec{A} = \vec{A}_1 + \vec{A}_2$ with $\nabla \times \vec{A}_1 = \vec{B}, \nabla \times \vec{A}_2 = 0$. The vector potential $\vec{A}_2$ is the additional AB flux field created by a solenoid [26] with the external magnetic field pointing in the z direction ($\vec{B} = B\hat{z}$). The vector potentials [27] are given by

$$\vec{A}_1 = \frac{\vec{B}e^{-\alpha r}}{1 - e^{-\alpha r}}\hat{\phi}, \quad \vec{A}_2 = \frac{\Phi_{AB}}{2\pi r}\hat{\phi}$$

$$\vec{A} = \vec{A}_1 + \vec{A}_2 = \left(\frac{\vec{B}e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{\Phi_{AB}}{2\pi r}\right)\hat{\phi} \quad (7)$$

The total wave function is given by

$$\psi_{nm}(r, \phi) = \frac{1}{\sqrt{2\pi r}} e^{im\phi} R_{nm}(r), \quad m = 0, \pm 1, \pm 2 \dots \quad (8)$$

where m is the magnetic quantum number. In this paper, we consider the condition for equal vector and scalar potentials such that Eq. (6) reduces to

$$\left\{c^2 \left(-i\hbar\nabla + \frac{e}{c}\vec{A}\right)^2 - (E_{nm}^2 - M_0^2c^4 - 2U(r)(E_{nm} + M_0c^2))\right\} \psi_{nm}(r, \phi) \quad (9)$$

In cylindrical coordinate, the respective Laplacian and Del operators are given as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \quad (10)$$

$$\vec{\nabla} = \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \phi} \hat{\phi} + \frac{\partial}{\partial z} \hat{z} \quad (11)$$

By inserting Eqs. (1), (7), (8), (10) and (11) into (9), we obtained the Schrodinger-type equation

$$\frac{d^2 R_{nm}(r)}{dr^2} + (\varepsilon_{nm} - U_{\text{eff}}(r, \omega_c, \xi)) R_{nm}(r) = 0 \quad (12)$$

where

$$\begin{aligned} \varepsilon_{nm} &= \frac{1}{\hbar^2 c^2} (E_{nm}^2 - M_0^2 c^4) \\ U_{\text{eff}}(r, \omega_c, \xi) &= \Lambda \left(A - \frac{B}{r} \right) \left(C - \frac{De^{-\alpha r}}{r} \right) \\ &+ \frac{M_0^2 \omega_c^2}{\hbar^2} \left(\frac{e^{-2\alpha r}}{(1 - e^{-\alpha r})^2} \right) \\ &+ \frac{2M_0 \omega_c}{\hbar} (m + \xi) \left(\frac{e^{-\alpha r}}{r(1 - e^{-\alpha r})} \right) + \frac{L_m}{r^2} \end{aligned} \quad (13)$$

The notations Λ and L_m in (13) are given by

$$\begin{aligned} \Lambda &= \frac{2(E_{nm} + M_0 c^2)}{\hbar^2 c^2} \\ L_m &= (m + \xi)^2 - \frac{1}{4} \end{aligned}$$

Also, in Eq. (13), $\omega_c = \frac{|e|\hbar B}{M_0 c}$ is the cyclotron frequency and $\xi = \frac{\Phi_{AB}}{\phi_0}$ is an integer having a flux quantum $\phi_0 = \frac{hc}{e}$.

2 Bound states solutions of the KGE and SE via the SWKB method

The SWKB approach [28] which is constructed from the lowest order WKB approximation and supersymmetry in quantum mechanics has been used to obtain the bound state solutions of the wave equations with many quantum potential models [29, 30]. The Schrodinger-type equation in (12) for the ground state wave function can be expressed as

$$-\frac{d^2 R_{0m}(r)}{dr^2} + U^{(1)}(r) R_{0m}(r) = 0 \quad (14)$$

where

$$U^{(1)}(r) = U_{\text{eff}}(r) - \varepsilon_{0m} \quad (15)$$

Using supersymmetry in quantum mechanics, we define the super-potential as

$$W(r) = -\frac{1}{R_{0m}} \frac{dR_{0m}}{dr} \quad (16)$$

which satisfies the Riccati equation

$$U^{(1)}(r) = W^2(r) - \frac{dW(r)}{dr} \quad (17)$$

With $U^{(1)}(r)$, we can define another potential $U^{(2)}(r)$

$$U^{(2)}(r) = W^2(r) + \frac{dW(r)}{dr} \quad (18)$$

The partner potentials $U^{(1)}(r)$ and $U^{(2)}(r)$ satisfy the shape invariance condition

$$U^{(2)}(r, a_2) = U^{(1)}(r, a_1) + R(a_2) \quad (19)$$

where a_1 is a set of parameters and a_2 is a set of parameters which depend on a_1 . Also, $R(a_2)$ is a parameter independent of r but shift the energy scale.

Equation (14) has a ground state of zero energy $\varepsilon_{0m}^{(1)} = 0$ which is the exact value. The ground state energy of $U^{(2)}(r)$ is not zero ($\varepsilon_{0m}^{(2)} \neq 0$). Supersymmetric quantum mechanics allows us to write the partner Hamiltonians

$$\begin{aligned} H^{(1)} &= -\frac{d^2}{dr^2} + U^{(1)}(r) = A^\dagger A \\ H^{(2)} &= -\frac{d^2}{dr^2} + U^{(2)}(r) = A A^\dagger \end{aligned} \quad (20)$$

The operators are defined as

$$\begin{aligned} A^\dagger &= -\frac{d}{dr} + W(r) \\ A &= \frac{d}{dr} + W(r) \end{aligned} \quad (21)$$

The relationships between the energy eigenvalues of the partner Hamiltonians are given by

$$\varepsilon_{nm}^{(2)} = \varepsilon_{nm}^{(1)} + \varepsilon_{0m} \quad (22)$$

By using Eqs. (15) and (17), the ground state energy can be obtained easily. However, to obtain the excited states using the SWKB method, we utilize the energy quantization rule [28–30]

$$\int_{r_1}^{r_2} dr \sqrt{\varepsilon_{nm}^{(1)} - W^2(r)} = n\pi, \quad n = 0, 1, 2, 3, \dots \quad (23)$$

Furthermore, we propose a super-potential of the form

$$W(r, \omega_c, \xi) = f - \frac{ge^{-\alpha r}}{1 - e^{-\alpha r}}, \quad f, g > 0 \quad (24)$$

Using Eqs. (15) and (17), we arrived at

$$U_{eff}(r, \omega_c, \xi) - \varepsilon_{0m}(\omega_c, \xi) = W^2(r, \omega_c, \xi) - \frac{dW(r, \omega_c, \xi)}{dr} \quad (25)$$

To solve (25), we applied the centrifugal approximation proposed by Greene and Aldrich [31]

$$\frac{1}{r} \approx \frac{\alpha}{1 - e^{-\alpha r}}, \quad \alpha \ll 1 \quad (26)$$

Substituting the super-potential, the effective potential with the centrifugal approximation into (25), and solving completely yields three equations:

$$\Lambda A(C + \alpha D) + \frac{M_0^2 \omega_c^2}{\hbar^2} - \varepsilon_{0m}(\omega_c, \xi) = (f + g)^2 \quad (27)$$

$$\frac{2\alpha M_0 \omega_c (m + \xi)}{\hbar} - \Lambda(A - \alpha B)(C + \alpha D) - \Lambda AC + 2\varepsilon_{0m}(\omega_c, \xi) = -2f^2 - 2gf - \alpha g \quad (28)$$

$$\Lambda(A - \alpha B)C + L_m \alpha^2 - \varepsilon_{0m}(\omega_c, \xi) = f^2 \quad (29)$$

Solving Eqs. (27)–(29) completely admits the ground state energy of the KGE

$$\varepsilon_{0m}(\omega_c, \xi) = \Gamma_0 - \frac{1}{4} \left(g + \frac{\Gamma_1}{g} \right)^2 \quad (30)$$

where

$$\Gamma_0 = \Lambda(A - \alpha B)C + \alpha^2 \left((m + \xi)^2 - \frac{1}{4} \right) \quad (31)$$

$$\Gamma_1 = \alpha^2 \left((m + \xi)^2 - \frac{1}{4} \right) - \frac{M_0^2 \omega_c^2}{\hbar^2} - \Lambda \alpha (BC + AD) \quad (32)$$

$$g = \alpha \left(\frac{1}{2} \pm \sqrt{\left[(m + \xi) + \frac{M_0 \omega_c}{\alpha \hbar} \right]^2 + \Lambda BD} \right) \quad (33)$$

To obtain the excited states energy spectra, we used the SWKB energy quantization rule given in Eq. (23)

$$\int_{r_1}^{r_2} dr \sqrt{\varepsilon_{nm}^{(1)}(\omega_c, \xi) - \left(f - \frac{ge^{-\alpha r}}{1 - e^{-\alpha r}} \right)^2} = n\pi \quad (34)$$

Using the coordinate transformation $s = 1 - 2e^{-\alpha r} \in (-1, 1)$ (34) turned out as

$$\sqrt{\frac{N}{\alpha^2}} \int_{-1}^1 \frac{ds}{1 - s^2} \sqrt{-s^2 + \Omega s - Q} = n\pi \quad (35)$$

where

$$Q = \frac{(f - g)^2 - \varepsilon_{nm}^{(1)}(\omega_c, \xi)}{N} \quad (36)$$

$$\Omega = \frac{2(g^2 - f^2) - 2\varepsilon_{nm}^{(1)}(\omega_c, \xi)}{N} \quad (37)$$

$$N = (f + g)^2 - \varepsilon_{nm}^{(1)}(\omega_c, \xi) \quad (38)$$

The solution to the integral in (35) is given [30] as

$$\int_{-1}^1 \frac{ds}{1 - s^2} \sqrt{-s^2 + \Omega s - Q} = \frac{\pi}{2} \left[2 - \sqrt{1 - \Omega + Q} - \sqrt{1 + \Omega + Q} \right] \quad (39)$$

Comparing Eqs. (35) and (39), the excited state energy spectra corresponding to $H^{(1)}$ are obtained as

$$\varepsilon_{nm}^{(1)}(\omega_c, \xi) = f^2 - \frac{1}{4} \left(n\alpha + g + \frac{f^2 - (f + g)^2}{n\alpha + g} \right)^2 \quad (40)$$

It is easy to see that for $n = 0$, the SWKB ground state energy is exact. Using the identity given in Eq. (22), the total energy of the system which corresponds to $H^{(2)}$ is obtained as

$$\varepsilon_{nm}^{(2)}(\omega_c, \xi) = E_0 - \frac{\alpha^2}{4} \left(n + \Delta + \frac{E_1}{n + \Delta} \right)^2 \quad (41)$$

where

$$E_0 = \Lambda(A - \alpha B)C + \alpha^2 \left((m + \xi)^2 - \frac{1}{4} \right) \quad (42)$$

$$E_1 = \left((m + \xi)^2 - \frac{1}{4} \right) - \frac{M_0^2 \omega_c^2}{\alpha^2 \hbar^2} - \frac{\Lambda(BC + AD)}{\alpha} \quad (43)$$

$$\Delta = \frac{1}{2} \pm \sqrt{\left[(m + \xi) + \frac{M_0 \omega_c}{\alpha \hbar} \right]^2 + \Lambda BD} \quad (44)$$

The corresponding Schrodinger equation energy spectra can be obtained by using the non-relativistic limits $\frac{E_{nm} + M_0 c^2}{\hbar^2 c^2} \rightarrow \frac{m_e}{\hbar^2}$, $\frac{E_{nm} - M_0 c^2}{\hbar^2 c^2} \rightarrow 2E_{nm}^{NR}$, $M_0 \rightarrow m_e$, where E_{nm}^{NR} is the non-relativistic energy and m_e is the effective mass

$$E_{nm}^{(NR)}(\omega_c, \xi) = \Upsilon_0 - \frac{\alpha^2 \hbar^2}{8m_e} \left(n + \delta + \frac{\Upsilon_1}{n + \delta} \right)^2 \quad (45)$$

where

$$\Upsilon_0 = (A - \alpha B)C + \frac{\alpha^2 \hbar^2}{2m_e} \left((m + \xi)^2 - \frac{1}{4} \right) \quad (46)$$

$$\Upsilon_1 = \left((m + \xi)^2 - \frac{1}{4} \right) - \frac{m_e^2 \omega_c^2}{\alpha^2 \hbar^2} - \frac{2m_e(BC + AD)}{\alpha \hbar^2} \quad (47)$$

$$\delta = \frac{1}{2} \pm \sqrt{\left[(m + \xi) + \frac{m_e \omega_c}{\alpha \hbar} \right]^2 + \frac{2m_e BD}{\hbar^2}} \quad (48)$$

Using Eq. (16) with the proposed super-potential, the ground state radial wave function can be expressed as

$$R_{0m}(r, \omega_c, \xi) = e^{-fr} (1 - e^{-\alpha r})^{\frac{a}{\alpha}} \quad (49)$$

Using the relation given in (8), the total wave function is obtained as

$$\begin{aligned} \psi_{0m}(r, \omega_c, \xi, \phi) &= \frac{N_{0m}}{\sqrt{2\pi r}} e^{-fr} (1 - e^{-\alpha r})^{\frac{a}{\alpha}} e^{im\phi}, \quad m \\ &= 0, \pm 1, \pm 2 \dots \end{aligned} \quad (50)$$

where $N_{0m}(\omega_c, \xi)$ is the normalization constant which can be obtained from the normalization condition

$$\frac{N_{0m}^2(\omega_c, \xi)}{2\pi} \int_0^{2\pi} d\phi (e^{-im\phi}) e^{im\phi} \int_0^\infty \frac{1}{r} e^{-2fr} (1 - e^{-\alpha r})^{\frac{2a}{\alpha}} r dr = 1 \quad (51)$$

Using the change of variable $s = e^{-\alpha r}$, the integral in (51) reduces to the familiar beta function integral

$$N_{0m}^2(\omega_c, \xi) \int_0^1 s^{\frac{2f}{\alpha}-1} (1-s)^{\frac{2a}{\alpha}} ds = 1 \quad (52)$$

Table 1 Energy spectra of the KGE in the absence and presence of magnetic and AB flux fields. We used the arbitrary parameters ($A = 2, B = 1, C = 2, D = 1, M_0 = 0.5, \alpha = 0.025, \hbar = 1, c = 1$)

State		E_{nm}			
m	n	$\omega_c = \xi = 0$	$\omega_c = 4, \xi = 0$	$\omega_c = 0, \xi = 4$	$\omega_c = \xi = 4$
0	0	- 0.4999824438	- 0.4999651097	- 0.4999827873	- 0.4997011114
	1	- 0.4999512032	- 0.4998277196	- 0.4998800206	- 0.4990479835
	2	- 0.4998812414	- 0.4995575351	- 0.4997125995	- 0.4982797241
	3	- 0.4997759148	- 0.4991592825	- 0.4994952536	- 0.4973999812
1	0	- 0.4999827019	- 0.4998966367	- 0.4999827950	- 0.4996390380
	1	- 0.4999160741	- 0.4996256243	- 0.4998749654	- 0.4988644278
	2	- 0.4998126666	- 0.4992265760	- 0.4996952140	- 0.4979785705
	3	- 0.4996740590	- 0.4987039215	- 0.4994598290	- 0.4969848853
- 1	0	- 0.4999827019	- 0.5000353373	- 0.4999827752	- 0.4997646978
	1	- 0.4999160741	- 0.5000348886	- 0.4998871514	- 0.4992359274
	2	- 0.4998126666	- 0.4998966367	- 0.4997355277	- 0.4985879401
	3	- 0.4996740590	- 0.4996256243	- 0.4995398452	- 0.4978246279

Table 2 Maximum non-relativistic vibrational energy in the absence and presence of magnetic and AB flux fields ($A = 2, B = 1, C = 2, D = 1, \hbar = 1, m_e = 0.5, \alpha = 0.025$)

	$E_{n_{\max}, -1}$	$E_{n_{\max}, 0}$	$E_{n_{\max}, 1}$
$\omega_c = \xi = 0$	3.950468750	3.949843750	3.950468750
$\omega_c = 4, \xi = 0$	3.950468750	3.949843750	3.950468750
$\omega_c = 0, \xi = 4$	3.955468750	3.959843750	3.965468750
$\omega_c = \xi = 4$	3.955468750	3.959843750	3.965468750

Performing the integration, we found the normalization constant such that the normalized ground state total wave function under magnetic and AB flux fields becomes

$$\begin{aligned} \psi_{0m}(r, \omega_c, \xi, \phi) &= \sqrt{\frac{\alpha \Gamma\left(\frac{2f}{\alpha} + \frac{2g}{\alpha} + 1\right)}{2\pi r \Gamma\left(\frac{2f}{\alpha}\right) \Gamma\left(\frac{2g}{\alpha} + 1\right)}} e^{-fr} (1 - e^{-\alpha r})^{\frac{a}{\alpha}} e^{im\phi} \end{aligned} \quad (53)$$

3 Discussion of results

In Table 1, in the absence of magnetic and AB flux fields, energy degeneracy is observed for the states with $m = 1$ and $m = -1$. The introduction of the magnetic field in the absence of the AB field removes the degeneracy; however under the AB flux field and in the absence

of the magnetic field, the observed energy overlaps are also removed. Furthermore, in the presence of the magnetic and AB flux fields, the degeneracy is observed to be removed and the eigenenergies are strongly bounded. This implies that both fields have a stronger influence on the relativistic energy of the system than the magnetic field alone or the AB field alone. Furthermore, for a quantum state with a particular value of m , the relativistic energies increase as n increases, regardless of the presence or absence of external fields. In Table 2, the presence of the magnetic field does not affect the value of the maximum non-relativistic vibrational energies of the system. The maximum vibrational energy of the system remains constant for all states. However, when the AB flux field is turned on, it causes

the maximum energy to shift upwards. Also, the AB flux effects remove the energy overlap. When both magnetic and AB flux fields are present, the energy of the system remains constants, suggesting that the Zeeman effects are not observed for the maximum vibrational energy. Our results support the theory put forward by Aharonov and Bohm [5] where a moving charge is changed by scalar and vector potentials which appear in the wave equations even in the absence of external electromagnetic fields [14].

In Fig. 1a, the non-relativistic energy is observed to increase and tend to a constant value as n increases. Also in the absence of magnetic and AB flux fields the energy eigenvalues overlap for the states $m = -1, 1$. In Fig. 1b, the introduction of the magnetic field causes

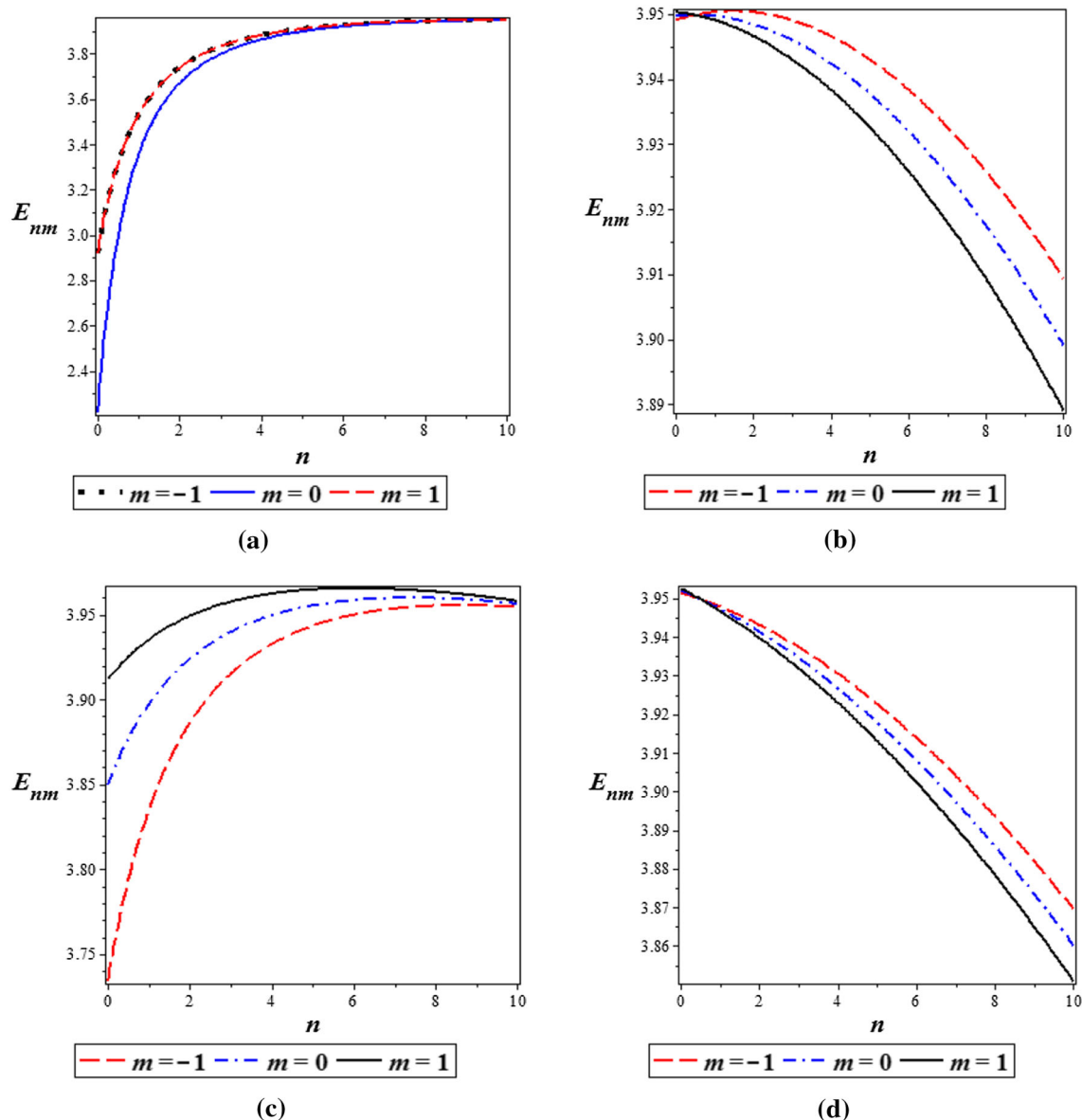


Fig. 1 Non-relativistic two-dimensional energy eigenvalues variations with principle quantum number in the absence and presence of magnetic and AB flux fields. We used the following arbitrary constants for the potential energy. $A = 2$, $B = 1$, $C = 2$, $D = 1$, $\hbar = 1$, $m_e = 0.5$, $\alpha = 0.025$. **a** $\omega_c = \xi = 0$, **b** $\omega_c = 4, \xi = 0$, **c** $\omega_c = 0, \xi = 4$, **d** $\omega_c = \xi = 4$

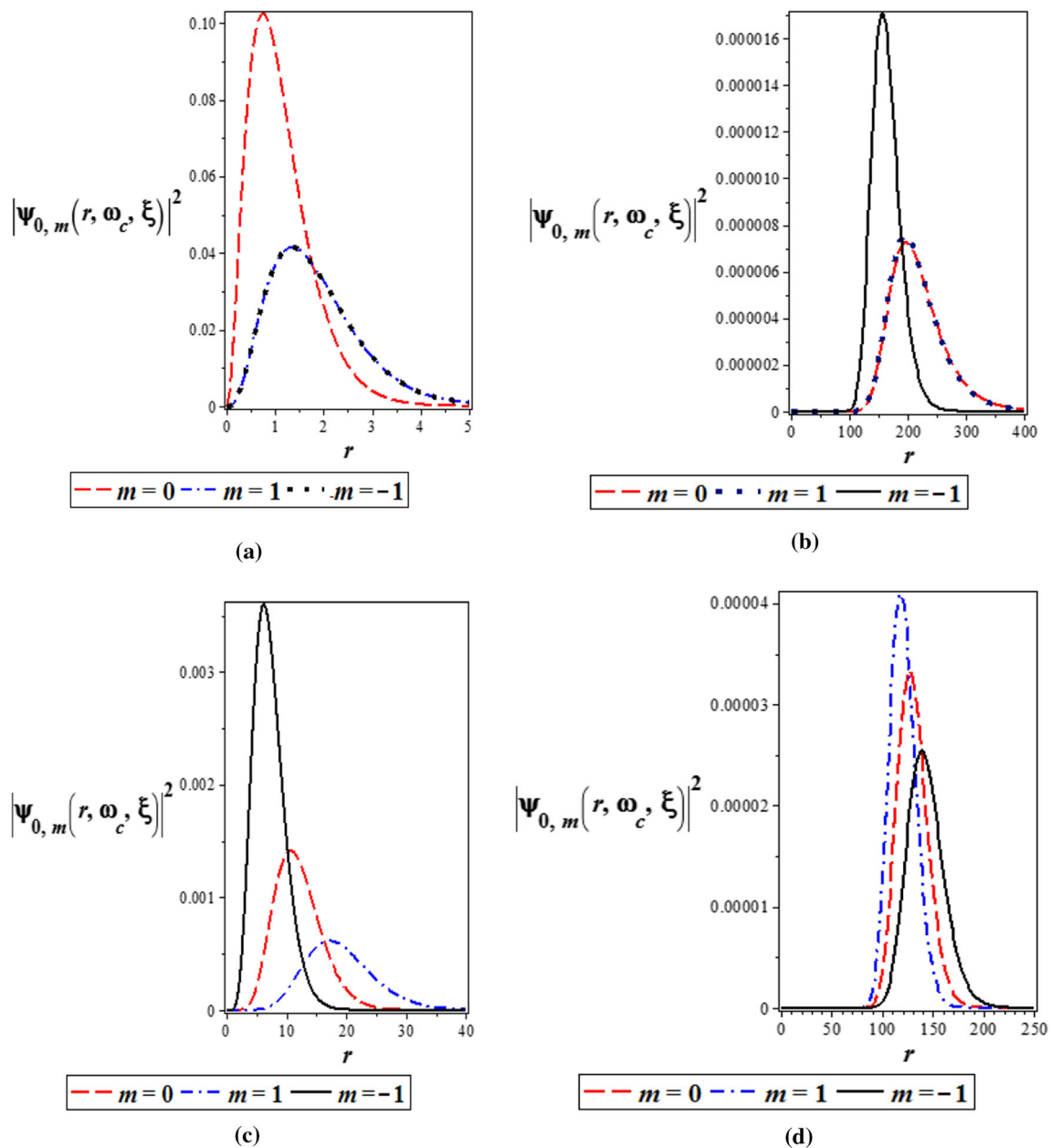


Fig. 2 Non-relativistic two-dimensional probability density variations with radial distance in the absence and presence of magnetic and AB flux fields. We used the following arbitrary constants for the potential energy. $A = 2$, $B = 1$, $C = 2$, $D = 1$, $\hbar = 1$, $m_e = 0.5$, $\alpha = 0.025$. **a** $\omega_c = \xi = 0$, **b** $\omega_c = 4, \xi = 0$, **c** $\omega_c = 0, \xi = 4$, **d** $\omega_c = \xi = 4$

the degeneracy to be lifted and the energy decreases as n increases. In Fig. 1c, the energy increases as n increases for all states as the AB flux field is introduced. In Fig. 1d, the energy decreases from maximum to lower states in the presence of both magnetic and AB flux fields. In Fig. 2a, the probability density function overlaps for the quantum states $m = -1, 1$ but gives a maximum peak for $m = 0$ when the magnetic and AB flux fields are turned off. In Fig. 2b–d, the introduction of the magnetic and or AB flux fields lifts the probability density function overlaps. Our results are

consistent with the results reported in the existing literature [8, 13] where the authors used different potential energy functions. Furthermore, the obtained closed-form relativistic energy reduces to the Schrodinger equation via the non-relativistic limits. Our potential reduces to the Screened Kratzer, Kratzer, Yukawa and Coulomb's potential where their corresponding energies can be obtained for both relativistic and non-relativistic regimes.

4 Conclusion

In this work, we studied the influence of the external magnetic field and AB flux fields under the two-dimensional relativistic and non-relativistic regimes with a quantum mechanical solvable potential model. We obtained the approximate normalized ground state wave function and also the energy spectra in closed form using the SWKB approach. The relativistic energy of a charged particle overlaps when external magnetic and AB flux fields are absent for the quantum states $m = -1, 1$ as n increases. The degeneracy is lifted when external fields are turned on. Our potential model reduces to the Screened Kratzer, Kratzer, Yukawa and Coulomb's potential as special cases and the corresponding normalized wave function and energy levels can be obtained for both relativistic and non-relativistic wave equations.

Author's contribution

E. Omugbe, I.J. Njoku and I.B. Okon wrote the first draft. U.S. Okorie, C.A. Onate and E. Omugbe carried out result confirmation and discussion. O. E. Osafire, A. Jahanshir and K.O. Suleman handled literature search and editing.

Data Availability Statement The data in this article were obtained from the analytical solutions presented. Therefore, no data were used in the article.

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