

Research

Existence and stability of equilibrium points in the perturbed R3BP with Stokes drag in the binary η –Cassiopeiae system

Jessica Mrumun Gyegwe¹ · Aguda Ekele Vincent²

Received: 2 August 2024 / Accepted: 5 May 2025

Published online: 14 May 2025

© The Author(s) 2025 **OPEN**

Abstract

This paper investigates the motion of an exoplanet in the gravitational field of the binary η –Cassiopeiae system under the effect of Coriolis, centrifugal and Stokes drag (SD) forces of the binary system. We consider the plane motion of the primaries in the case where η –Cassiopeiae A and B are triaxial and oblate spheroidal bodies, respectively. In particular, we examine the existence, position, and nature of motion around the equilibrium points under the influence of the system parameters. It is found that under constant SD effect, collinear equilibrium points cease to exist numerically, but there are in the absence of the perturbing force. By a numerical investigation, it is found that there may be three, five, seven, nine or eleven non-collinear equilibria, depending on the values of the system parameters. Additionally, the positions of these points depend on all the system parameters except small perturbation in the Coriolis force. The mathematical model of this problem has been examined for the location and stability of the five non-collinear equilibrium points. A numerical exploration is performed using the binary η –Cassiopeiae system to compute the position of the equilibrium points and their stability. The positions of these equilibrium points are affected by the presence of perturbations, since they are deviated from their classical positions. The linear stability of the equilibrium points under the combined influence of the mass ratio and the perturbing forces is analyzed, and it is found that all the equilibrium points are unstable except L_4 and L_5 , which are stable for certain parameter values.

Keywords R3BP · Perturbations · Equilibrium points · Stokes drag · Linear stability

1 Introduction

The restricted three-body problem (R3BP) is one of the most important subjects of study in space dynamics. It is the description of the motion of a tiny mass body (third body) within a system consisting of two massive masses (the primaries), which orbit in circular or elliptical paths around their shared center of mass. The mass of the third body is insignificant in comparison to that of the primary bodies, so its influence on their motion is negligible. The first and most investigated version of this problem is the so-called circular restricted three-body problem (hereafter CR3BP). In the classical R3BP, there are five equilibrium or Lagrangian points. There are three collinear points on the x –axis that were discovered by Euler around 1750 and are denoted L_1, L_2, L_3 and there are two equilateral points discovered by Lagrange around 1760 and are denoted L_4, L_5 , and form equilateral triangles with the primaries, see [1]. The collinear equilibrium

✉ Jessica Mrumun Gyegwe, jessica.gyegwe@fulokoja.edu.ng; Aguda Ekele Vincent, vincentekele@yahoo.com | ¹Department of Mathematics, Federal University Lokoja, Lokoja, Kogi, Nigeria. ²Department of Mathematics, Nigeria Maritime University, Okerenkoko, Delta, Nigeria.



points are unstable whereas the triangular points are stable when the mass ratio μ of the primaries is below the critical threshold $\mu_0 \cong 0.03852090$ [2].

A Binary star system is composed of two stars that are bound gravitationally and move in circular or elliptical orbits around their common barycenter. The more massive of the two stars is called the primary star, while the less massive star is the secondary star. The situation whereby both stars in a binary system are larger than the planet depicts a good example of a R3BP (see, e.g., [2–8]). Thus, the equations of motion of the binary stars can be calculated independently before calculating the motion of the planet. In order to comprehend and analyze the dynamical behavior of the infinitesimal body or test particle as the case maybe in the R3BP, many researchers have considered perturbing the R3BP. Such considerations include the perturbations in the Coriolis and centrifugal forces. These two forces are called apparent or fictitious forces. From the physical point of view, the centrifugal force pulls all objects outwards from the axis of planetary rotation, while the Coriolis force acts on all moving bodies in a rotating frame of reference. Szebehely [9] established that the Coriolis force is a stabilizing force for point masses when keeping the centrifugal force constant. This work was extended by Subbarao and Sharma [10] and observed that when the primary body is taken to be an oblate spheroid and the centrifugal force is no longer a constant force, then the Coriolis force is not always a stabilizing force for the triangular equilibrium points. Also, Bhatnager and Hallan [11] investigated the joint effects of Coriolis and the centrifugal forces in the circular restricted problem of three bodies. Furthermore, Abdulraheem and Singh [12] observed that Coriolis force has a stabilizing tendency, while the centrifugal force, radiation pressure, and oblateness of the primaries have destabilizing effects, and this is in agreement with the results presented in [13–15]. In the case of three oblate bodies, Abouelmagd and Guirao [16] presented an analytical study on the perturbed planar restricted three-body problem. They explored the existence of equilibrium points and their linear stability under the effects of perturbations in Coriolis and centrifugal forces. For perturbations in the shape of the primaries of the R3BP, particularly when the primaries are spheroids, we may cite [13, 17–19]. Furthermore, when deviations from the sphericity of the primaries of the CR3BP are considered, it is also known that the system possesses additional equilibrium points to the classical collinear and triangular Lagrange points. In a recent work, Saeed and Zotos [20] obtained between three to eleven equilibrium points when one of the primaries is a triaxial oblate body. They also showed that all the equilibrium points are unstable. Also, Alrebdi et al. [21] obtained three to thirteen equilibrium points in the case where one of the primaries is a triaxial prolate ellipsoid (body). It was established that linearly stable and unstable points coexist. The study of these issues enables us to get real and accurate data about the dynamical features of the system. For example, Oberti and Vienne [22] showed that the addition of oblateness effect leads to improved approximations of real orbits of certain satellites in the Solar System.

The Stokes drag is a dissipative force. It is a force loss brought on by spherical mass moving through a viscous fluid, such as gas. This force is a linear function of the relative velocity of the particle pertaining to the gas. According to Jancart and Lemaitre [23], this force is represented as:

$$F = -C V r = -C (y - \alpha(\theta \times X)),$$

where C is a constant which depends on the physical properties of the gas and on the density and the size of the particles. $\mathbf{x}$, $\mathbf{y}$ are the Cartesian heliocentric coordinates and the Cartesian velocity vector, respectively. The parameter α is the ratio of gas velocity over Keplerian velocity while θ is the circular angular velocity at that point. Protoplanetary disks are thought to evolve into planetary systems based on the nebular hypothesis of solar system formation. Thus, in the framework of protoplanetary nebula, Stokes drag is used to illustrate the force loss acting on a particle moving in an assumed circular motion around a star. This dissipative force is modeled in synodic coordinates as (see, e.g., [24–26]):

$$(F_x, F_y) = -k(\dot{x} - y + \alpha\Omega_y, \dot{y} + x - \alpha\Omega_x), \quad (1)$$

where the dissipative constant is $k \in (0, 1)$ and it in turn depends on the viscosity of the gas and the particle's radius and mass (see, e.g., Beauge and Ferraz-Mello [27]). Also, the Keplerian angular velocity $\Omega = \Omega(r) = r^{-\frac{3}{2}}$ is at a distance $r = (x^2 + y^2)^{\frac{1}{2}}$ from the synodic frame's origin. The ratio of the gas and Keplerian velocities is denoted by $\alpha \in (0, 1)$ (see Murray [24]). Furthermore, Murray [28] studied the dynamical effects of three kinds of drag forces: the nebular drag, gas drag, and the Poynting-Robertson (P-R) drag in the CR3BP. They found that the locations of the collinear points are not on the axis joining the primaries even as the displaced triangular points L_4 and L_5 are asymptotically stable for certain classes of drag forces. Later on, this problem has been widely studied by many researchers. Jain and Aggarwal [29] examined the effect of Stokes drag force (SDF) on the existence and stability of equilibrium points in the R3BP. They asserted that only two non-collinear equilibrium points exist. These points are unstable. For the same variant where the secondary body is taken to be an oblate spheroid, Jain and Aggarwal [30] obtained similar results as [29]. In a similar line of inquiry,

Vincent and Perdiou [31] considered the existence and stability of equilibrium points under the effects of Stokes and P–R drags terms. They asserted that under the constant effects of P–R and/or Stokes drags, collinear equilibrium points disappear, but these points are there when these drag forces are absent. They discovered five equilibrium points that are non-collinear while the equilibria L_4 and L_5 are not symmetrical with respect to the Ox axis. They also demonstrated that these non-collinear points are unstable under the consequence of both kinds of drag forces except the equilibria L_4 and L_5 that are stable only for Stokes drag effect. In the literature several authors provided dynamical analyses in the context of the planar CR3BP for dissipation forces such as the Solar wind and/or P–R drags (see, e.g., [32–35]). Arif and Sagar [36] studied the location and stability of equilibrium points in the R3BP when the bigger primary is modeled as a uniform circular disc. They found six collinear and two non-collinear equilibrium points and observed that five equilibrium points are unstable while three are stable for the values of parameters μ and α . They noticed for the zero velocity curves; the outer ovals expand while the inner ovals slightly shrink as the value of the Jacobian constant C increases.

In their study on the bifurcation analysis and periodic solutions of the HD 191408 system with triaxial and radiative perturbations, Gao and Wang [37] discussed the relationship between the geometric configuration of the mathematical model with respect to the third body and the Jacobian constant. Recently, in [38], the effect of radiation pressure force on the formation of nominal halo orbit around the inner collinear point was investigated in the CR3BP with Sun–Earth system. Recent research has rekindled interest in understanding the dynamics of particles in the vicinity of asteroids or other minor celestial bodies with respect to the Earth, other planets, natural satellites and other massive celestial bodies in the Solar system. Through both an analytical approach and a numerical treatment, in [39], a new approach for solving dynamical equations of motion for the trapped motion of an infinitesimal third body in case of the elliptic restricted three-body problem (ER3BP) was presented. Their work further refines the understanding of the complex dynamics surrounding the equilibrium points and their stability. The suggested approach is particularly important in celestial mechanics, especially for optimizing the maneuvers of spacecraft in the vicinity of the barycenter in a frame of the ER3BP.

In this paper, the triaxiality factors of the primary body are of small quantities, and therefore we choose arbitrary values that are far less than one (see, e.g., Vincent et al. [8], Singh [40], Elshaboury et al. [19], and references therein). The methodology used in this paper to carry out the study of the location and linear stability of the fixed points of the system consists in solving analytically and numerically the dynamical equations that involve the parameters of the infinitesimal particle in the binary system under study.

The current study extends the research performed by Singh and Begha [14]. In particular, we present an analysis of the dynamical behavior of an exoplanet in the gravitational field of the binary η –Cassiopeia system under the effect of Coriolis, centrifugal and Stokes drag forces together with oblateness of the smaller primary body and triaxiality of the bigger primary body. While the initially formulated model effectively described the dynamics of a test body influenced by non-sphericity of the primaries together with Coriolis and centrifugal forces, it did not account for the influence of Stokes drag force, which is capable of altering orbital angular momentum and energy. The motivation for this extension arises from the increasing need for higher quality systems where Stokes drag force significantly affects the motion of small bodies, such as spacecraft, dust grains, comets, and asteroids moving around a star. In planetary dynamics, there are certain cases where interactions with gas particle can significantly affect the motion of solid particles. Two major examples of this phenomenon are the interactions of planetesimal with the protoplanetary disk during the formation of solar system and the orbital decay of ring particles as a result of drag caused by extended planetary atmospheres. This contribution is significant because it provides a more comprehensive framework for analyzing the dynamics of real physical systems, thereby improving the accuracy of trajectory design in deep space missions. An interesting result of this study is that, under constant effect of Stokes drag force, collinear equilibrium points cease to exist, while various non-collinear equilibrium points due to these varying parameters are no longer symmetrical with respect to the x –axis. This makes a qualitative difference in the dynamics of the model. These points and their stability are among the most characteristics features of any dynamical system not only due to their immediate applications in real systems but also due to the fact that these points are crucial for understanding the stability of orbits and for planning space missions. Eta Cassiopeiae (η –Cassiopeia, abbreviated Eta Cas, η –Cas) is a binary system in the northern constellation of Cassiopeia. Its binary nature was first discovered by William Herschel in August 1779 ([41]). The two components are designated Eta Cassiopeiae A and B.

The results obtained in this study may help to analyze more generalized problems of few bodies under the effects of radiation pressure, Poynting–Robertson drag, dark matter halo effect, among others (see, e.g., [42–45]).

In a recent paper, Elpe [46] has published seminal article where he argued against using of such perturbative effects such as perturbations in the Coriolis and/or centrifugal forces in the R3BP and various extensions of the three-body problem that include more than two finite bodies. This wrong hypothesis caused by the local not global idea: only the

sub-system is considered, not the whole system. If only sub-system is considered, Newton's third law can not be assumed. External force comes without any interaction. In some studies mentioned above, this local idea of sub-system has been used, especially for the mathematical point of view.

This paper is structured as follows: In Sect. 2, the materials and methods for the dynamical system are given. Section 3 presents the numerical application using the physical data of the η -Cassiopeia binary system regarding equilibrium points and nature of motion around these points. Finally, Sect. 4 summarizes the discussion and conclusion of our study.

2 Materials and methods

2.1 Equations of motion

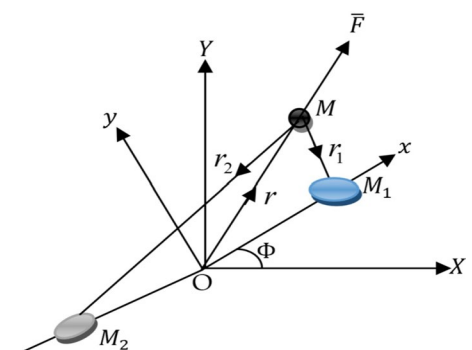
We investigate the motion of an exoplanet of negligible mass m under the action of two finite primary bodies, M_1 and M_2 , called the primaries (bigger primary and smaller primary, respectively), with masses m_1 and m_2 , respectively. We assume that the masses of the two main bodies which orbit in circular paths around their shared center of mass are large enough in comparison to the infinitesimal body (M) of mass m , such that the gravitational effect of m is negligible on the motion of the primaries. We let the bigger η -Cassiopeiae A body to be triaxial with mass m_1 and that of smaller η -Cassiopeiae B body be an oblate spheroid with mass m_2 such that $m_1 > m_2$. The X -axis is taken to be the line joining M_1 and M_2 with the origin being their barycenter O , and the Y -axis is the line passing through O and perpendicular to OX , lying in the plane of motion of M_1 and M_2 . Let r_1 and r_2 be the distances between M and the primary bodies M_1 and M_2 respectively, while r is the distance of infinitesimal body from the origin. We normalize the units such that the sum of the masses and the separation between the bodies are both equal to one and the time is defined such that the Gaussian constant of gravitation is equal to one, as well. We make use of the synodic coordinate system, $Oxyz$ which is initially concurrent with the inertial reference frame, $OXYZ$ and rotating with an angular velocity Φ about the Z -axis (the z -axis is concurrent with the Z -axis). And let $\bar{F}$ be the SDF. All of these details are captured in Fig. 1.

The equations governing the motion of the exoplanet under the influence of dissipative force and non-sphericity of the primaries are written in the form ([14, 29–31]):

$$\begin{aligned} \ddot{x} - 2n\dot{y} = & n^2x - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3} - \frac{3(1-\mu)(x-\mu)(2\delta_1-\delta_2)}{2r_1^5} - \frac{3A_2\mu(x+1-\mu)}{2r_2^5} \\ & + \frac{15(1-\mu)(x-\mu)(\delta_1-\delta_2)y^2}{2r_1^7} - k\left(\dot{x} - y - \frac{3\alpha y}{2r_1^{\frac{7}{2}}}\right), \\ \ddot{y} - 2n\dot{x} = & n^2y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} - \frac{3(1-\mu)(4\delta_1-3\delta_2)y}{2r_1^5} - \frac{3A_2\mu y}{2r_2^5} \\ & + \frac{15(1-\mu)(\delta_1-\delta_2)y^3}{2r_1^7} - k\left(x + \dot{y} + \frac{3\alpha x}{2r_1^{\frac{7}{2}}}\right), \end{aligned} \quad (2)$$

where

Fig. 1 The configuration of the perturbed R3BP with SDF and a triaxial primary as well as an oblate secondary



$$r_1 = [(x - \mu)^2 + y^2]^{\frac{1}{2}}, r_2 = [(x + 1 - \mu)^2 + y^2]^{\frac{1}{2}}. \quad (3)$$

The dimensionless quantities δ_1 and δ_2 are the so-called triaxiality parameters, which are explicitly given in terms of the semi-major axes of the triaxial bigger body (a, b and c along the x, y and z -axis respectively) as $\delta_1 = \frac{a^2 - c^2}{5R^2} \ll 1$, $\delta_2 = \frac{b^2 - c^2}{5R^2} \ll 1$. Also, A_2 is the oblateness of the smaller primary which is defined as $0 \leq A_2 = \frac{A_E^2 - A_P^2}{5R^2} \ll 1$, where A_E and A_P are the equatorial and polar radii, respectively, of the oblate primary and R is the separation of both primaries.

The mean motion of the exoplanet taking into cognizance the shape of the primaries is denoted by n and is expressed as

$$n^2 = 1 + \frac{3}{2}[(2\delta_1 - \delta_2) + A_2] \quad (4)$$

The mass parameter μ gives the ratio of the mass of the secondary star to the sum of the masses of both stars. That is, $\mu = \frac{m_2}{(m_1 + m_2)} \in (0, \frac{1}{2}]$, with $m_1 > m_2$. The bigger and smaller primaries are located at the positions $(\mu, 0)$ and $(\mu - 1, 0)$, correspondingly (see [2]).

Next, the perturbations in the Coriolis and centrifugal forces are introduced with the aid of the parameters v and ω with $v = \omega = 1$ being their unperturbed value. Consequently, Eq. (2) becomes

$$\ddot{x} - 2v n \dot{y} = U_x + F_x, \ddot{y} + 2v n \dot{x} = U_y + F_y, \quad (5)$$

where

$$U_x = n^2 \omega x - \frac{(1 - \mu)(x - \mu)}{r_1^3} \left[1 + \frac{3(2\delta_1 - \delta_2)}{2r_1^2} - \frac{15(\delta_1 - \delta_2)y^2}{2r_1^4} \right] - \frac{\mu(x + 1 - \mu)}{r_2^3} \left[1 + \frac{3A_2}{2r_2^2} \right],$$

$$U_y = y \left\{ n^2 \omega - \frac{(1 - \mu)}{r_1^3} \left[1 + \frac{3(4\delta_1 - 3\delta_2)}{2r_1^2} - \frac{15(\delta_1 - \delta_2)y^2}{2r_1^4} \right] - \frac{\mu}{r_2^3} \left[1 + \frac{3A_2}{2r_2^2} \right] \right\}, \quad (6)$$

with

$$U = \frac{n^2}{2} \omega (x^2 + y^2) + \frac{(1 - \mu)}{r_1} + \frac{\mu}{r_2} + \frac{(1 - \mu)(2\delta_1 - \delta_2)}{2r_1^3} - \frac{3(1 - \mu)(\delta_1 - \delta_2)y^2}{2r_1^5} + \frac{\mu A_2}{2r_2^3}, \quad (7)$$

$$F_x = -k \left(\dot{x} - y - \frac{3\alpha y}{2r^{\frac{7}{2}}} \right), F_y = -k \left(x + \dot{y} + \frac{3\alpha x}{2r^{\frac{7}{2}}} \right).$$

The perturbed value of v and ω are given as $v = 1 + \varepsilon, \omega = 1 + \varepsilon'$, $|\varepsilon|, |\varepsilon'| \ll 1$ where $\varepsilon, \varepsilon'$ represent the perturbations in the Coriolis and centrifugal forces, respectively. It is worth noticing that small perturbation in the Coriolis force (v) has no effect on the position of the equilibrium points as Eqs. (5) and (6) are independent of the parameter v .

The physical parameters of the binary η -Cassiopeia system are shown in Table 1 (see [41]).

2.2 Existence and location of the equilibrium points

The exoplanet experiences a state of rest at specific points when there is a balance in the centrifugal force and the gravitational attraction of the binary η -Cassiopeia system. Thus, equilibrium points (EPs) are obtained when the velocity and acceleration components of the exoplanet are zero. That is, when $\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0$. As such, Eqns. (5) and (6) become

Table 1 Numerical data for the binary η -Cassiopeia system

Binary system	$M_1(M_{sun})$	$M_2(M_{sun})$	Mass ratio (μ)
η Cassiopeia	1.11	0.6	0.35088

$$n^2\omega x - \frac{(1-\mu)(x-\mu)}{r_1^3} \left[1 + \frac{3(2\delta_1 - \delta_2)}{2r_1^2} - \frac{15(\delta_1 - \delta_2)y^2}{2r_1^4} \right] - \frac{\mu(x+1-\mu)}{r_2^3} \left[1 + \frac{3A_2}{2r_2^2} \right] + ky \left[1 + \frac{3}{2}\alpha(x^2 + y^2)^{-\frac{7}{2}} \right] = 0, \quad (8)$$

$$y \left\{ n^2\omega - \frac{(1-\mu)}{r_1^3} \left[1 + \frac{3(4\delta_1 - 3\delta_2)}{2r_1^2} - \frac{15(\delta_1 - \delta_2)y^2}{2r_1^4} \right] - \frac{\mu}{r_2^3} \left[1 + \frac{3A_2}{2r_2^2} \right] \right\} - kx \left[1 + \frac{3}{2}\alpha(x^2 + y^2)^{-\frac{7}{2}} \right] = 0. \quad (9)$$

It is worth noting that in the case where $\delta_1 = \delta_2 = A_2 = k = 0$ and $v = \omega = 1$, there are five coplanar equilibrium points [2]. Three of them being on the x -axis and are called collinear points (L_1 , L_2 , and L_3); and the other two (L_4 and L_5) are out of the x -axis and are called triangular equilibrium points of the problem. Specifically, points for which $y = 0$ are called collinear EPs and are situated on the line connecting the primaries; the x -axis of the synodic system while points for which $y \neq 0$ are called triangular (non-collinear) EPs and lie off the x -axis of the synodic system. In the present case, we will realize that the existence and number of the equilibrium points of the problem depend on the triaxiality of the primary as well as SDF of the system. Furthermore, the inclusion of dissipative forces, such as Stokes drag force from the binary system, adds further complexity to the system, making the characteristics of the equilibrium points highly sensitive to the Stokes drag coefficient.

The existence and location of the equilibrium points can be partly determined by studying analytically Eqs. (8) and (9). However, we observe that for $y = 0$, this solution (collinear) does not verify Eq. (9) due to the existence of the dissipative term defined by the Stokes drag. That is, it is not solution of Eqs. (8) and (9) (arising from Eqs. (5) and (6)). For collinear EPs Eq. (9), for $y = 0$, gives

$$-kx \left[1 + \frac{3}{2}\alpha x^{-7} \right] = 0, k = 0.$$

In the case where $k \neq 0$, meaning that the primaries are under the influence of dissipative force, equilibrium points of the form $(x_0, 0)$ do not exist analytically and of course, numerically for this system. As a result, the equilibrium points on the orbital plane are positioned on the Oxy plane, necessitating the solution of the Eqs. (8) and (9) for $(x, y) \neq (0, 0)$. This fact applies to similar studies where dynamical effect of different kinds of dissipation in the CRTBP were discussed (see, e.g., [28, 31, 33], and references quoted therein). It is obvious that collinear equilibrium solution does not exist under constant Stokes drag term. This is no longer true when the drag force is neglected. Therefore, we conclude that under the effect of Stokes drag of the primaries, there are no equilibrium points that lie exactly on the x -axis, called collinear EPs. So, we restrict our analysis to the non-collinear EPs.

The locations of the non-collinear EPs are the solutions of Eqs. (8) and (9) when $x, y \neq 0$. We remark that in the case where $k \neq 0$, perturbations method (analytical or semi-analytical solutions) is not enough to determine the exact locations of the equilibria due to the asymmetry of the problem (see, e.g., Murray [28], Vincent and Perdiou [31]). This means that, we may have $x(L_4) \neq x(L_5)$ and $y(L_4) \neq -y(L_5)$ where the equilibria exist and that the stability is close to L_4 is different from the stability close to L_5 (see, e.g., [29, 31], and references therein). The second fact is that due to highly nonlinear nature of the Hamiltonian system, there is an extra-difficulty not only to find the general analytical solutions for the system of Eqs. (8) and (9), but even to discuss the existence and the number of equilibria for every set of parameter values. Consequently, we resort to a numerical solution of this model problem. This fact applies to similar studies, where numerical methods are used for determining the equilibrium points of a system (see, e.g., [32–35]). One of the easiest ways of solving a nonlinear system of equations is by deploying the classical Newton–Raphson (NR) iterative method. To date, the NR method has been successfully applied to a wide variety of nonlinear dynamical systems for the determination of equilibrium points (see, e.g., [20, 31, 35]).

3 Linear stability of the equilibrium points

To examine the stability of the non-collinear equilibrium points, we place the infinitesimal mass at one of the equilibrium points and give it a small velocity. The point is stable for the oscillatory solutions with small amplitude and unstable for exponentially diverging solutions.

Let the position of an equilibrium point be denoted by (x_0, y_0) , and consider a small displacement (ξ, η) from the point such that

$$x = x_0 + \xi, y = y_0 + \eta \quad (10)$$

Substituting these values into Eqs. (5) and (6), we get the variational equations:

$$\begin{aligned} \ddot{\xi} - 2nv\dot{\eta} &= \xi(U_{xx}^0 + F_{xx}^0) + \eta(U_{xy}^0 + F_{xy}^0) + F_{xx}^0\dot{\xi}, \\ \ddot{\eta} + 2nv\dot{\xi} &= \xi(U_{yx}^0 + F_{yx}^0) + \eta(U_{yy}^0 + F_{yy}^0) + F_{yy}^0\dot{\eta}, \end{aligned} \quad (11)$$

where superscript (0) represents the value evaluated at the equilibrium point (x_0, y_0) , the subscripts represent the second-order derivatives while the dots indicate the derivatives with respect to time t .

Suppose $\xi = Ae^{\lambda t}$ and $\eta = Be^{\lambda t}$ are the trial solutions of Eqs. (11), where A, B and λ are parameters to be determined. Then, by using these values of the trial solutions in Eqs. (11), we get

$$\begin{aligned} (\lambda^2 - \lambda F_{xx}^0 - (U_{xx}^0 + F_{xx}^0))A - (2nv\lambda + (U_{xy}^0 + F_{xy}^0))B &= 0, \\ (2nv\lambda - (U_{yx}^0 + F_{yx}^0))A + (\lambda^2 - \lambda F_{yy}^0 - (U_{yy}^0 + F_{yy}^0))B &= 0 \end{aligned} \quad (12)$$

Now, for the nontrivial solution the determinant of the coefficients matrix of the above system must be zero, i.e

$$D = \begin{vmatrix} (\lambda^2 - \lambda F_{xx}^0 - S_{1xx}^0) & -(2nv\lambda + S_{1xy}^0) \\ (2nv\lambda - S_{2yx}^0) & (\lambda^2 - \lambda F_{yy}^0 - S_{2yy}^0) \end{vmatrix} = 0 \quad (13)$$

where we have abbreviated

$$S_{1xx}^0 = (U_{xx}^0 + F_{xx}^0), S_{1xy}^0 = (U_{xy}^0 + F_{xy}^0), S_{2yx}^0 = (U_{yx}^0 + F_{yx}^0), S_{2yy}^0 = (U_{yy}^0 + F_{yy}^0)$$

Simplifying Eq. (13), we obtain the characteristic polynomial corresponding to the system (12) as ([35, 47]):

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0, \quad (14)$$

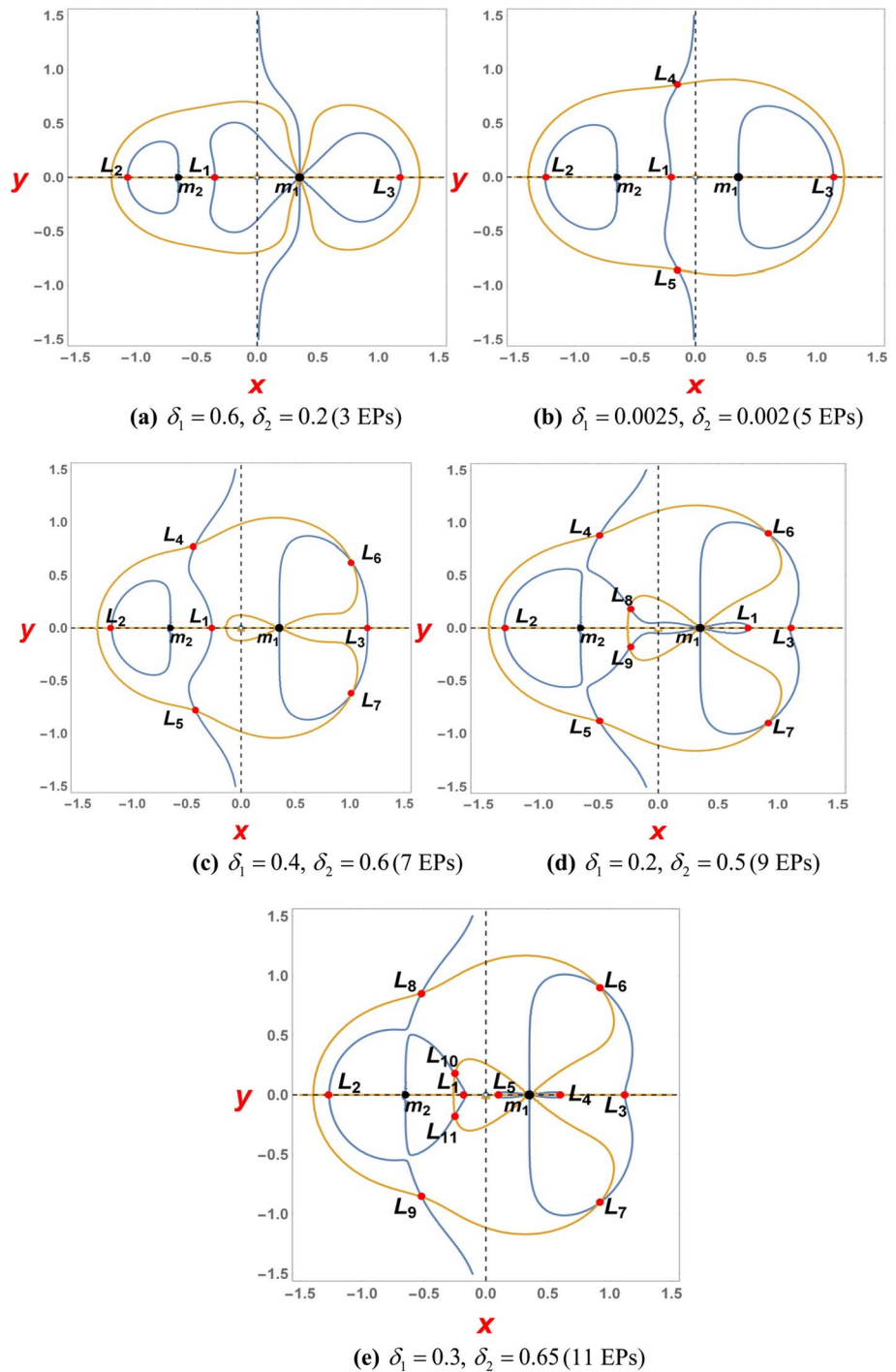
where

$$\begin{aligned} a &= -(F_{xx}^0 + F_{yy}^0), \\ b &= 4n^2v^2 + F_{xx}^0F_{yy}^0 - (U_{xx}^0 + F_{xx}^0) - (U_{yy}^0 + F_{yy}^0), \\ c &= F_{xx}^0(U_{yy}^0 + F_{yy}^0) + F_{yy}^0(U_{xx}^0 + F_{xx}^0) + 2nv(U_{xy}^0 + F_{xy}^0) - 2nv(U_{yx}^0 + F_{yx}^0), \\ d &= (U_{xx}^0 + F_{xx}^0)(U_{yy}^0 + F_{yy}^0) - (U_{xy}^0 + F_{xy}^0)(U_{yx}^0 + F_{yx}^0). \end{aligned} \quad (15)$$

Equation (14) is a fourth-degree polynomial in λ , and the resulting eigenvalues determine the linear stability or instability of the corresponding EPs. An equilibrium point (x_0, y_0) is considered linearly stable if all four roots of the characteristic equation are purely imaginary or have complex roots with each of them having negative real parts; otherwise, the point is deemed unstable.

By calculating the second-order derivatives of the modified potential-like function at the equilibrium point, we obtain the following:

Fig. 2 The positions of the non-collinear equilibrium points, $L_i, i = 1, 2, \dots, 11$ through the intersections of Eq. (8) (blue curve) and Eq. (9) (brown curve) for the binary η -Cassiopeiae system, when $\mu = 0.35088, k = 10^{-5}, \alpha = 0.05, A_2 = 0.01, \omega = 1.01$ for various values of δ_1 and δ_2 : (a) $\delta_1 = 0.6, \delta_2 = 0.2$ (3 EPs), (b) $\delta_1 = 0.0025, \delta_2 = 0.002$ (5 EPs), (c) $\delta_1 = 0.4, \delta_2 = 0.6$, (d) $\delta_1 = 0.2, \delta_2 = 0.5$; and (e) $\delta_1 = 0.3, \delta_2 = 0.65$. Red dots represent the equilibrium points while black dots mark the centers of the primary bodies



$$U_{xx}^0 = n^2 \omega - \frac{(1-\mu)}{r_{10}^3} - \frac{\mu}{r_{20}^3} + \frac{3(1-\mu)(x_0-\mu)^2}{r_{10}^5} - \frac{3\mu A_2}{2r_{20}^5} + \frac{3\mu(x_0-\mu+1)^2}{r_{20}^5} - \frac{3(2\delta_1-\delta_2)(1-\mu)}{2r_{10}^5} \\ + \frac{15(2\delta_1-\delta_2)(1-\mu)(x_0-\mu)^2}{2r_{10}^7} + \frac{15\mu A_2(x_0-\mu+1)^2}{2r_{20}^7} + \frac{15(\delta_1-\delta_2)y_0^2(1-\mu)}{2r_{10}^7} - \frac{105(\delta_1-\delta_2)y_0^2(1-\mu)(x_0-\mu)^2}{2r_{10}^9},$$

$$U_{yy}^0 = n^2 \omega - \frac{(1-\mu)}{r_{10}^3} - \frac{\mu}{r_{20}^3} + \frac{3(1-\mu)y_0^2}{r_{10}^5} - \frac{3(\delta_1 - \delta_2)(1-\mu)}{r_{10}^5} - \frac{3(2\delta_1 - \delta_2)(1-\mu)}{2r_{10}^5} + \frac{3\mu y_0^2}{r_{20}^5} - \frac{3A_2\mu}{2r_{20}^5} \\ + \frac{75(\delta_1 - \delta_2)y_0^2(1-\mu)}{2r_{10}^7} + \frac{15(2\delta_1 - \delta_2)(1-\mu)y_0^2}{2r_{10}^7} + \frac{15A_3\mu y_0^2}{2r_{20}^7} - \frac{105(\delta_1 - \delta_2)(1-\mu)y_0^4}{2r_{10}^9},$$

$$U_{xy}^0 = \frac{3y_0(1-\mu)(x_0 - \mu)}{r_{10}^5} + \frac{3\mu y_0(x_0 - \mu + 1)}{r_{20}^5} + \frac{15(\delta_1 - \delta_2)y_0(1-\mu)(x_0 - \mu)}{r_{10}^7} + \frac{15(2\delta_1 - \delta_2)y_0(1-\mu)(x_0 - \mu)}{2r_{10}^7} \\ + \frac{15A_2y_0\mu(x_0 - \mu + 1)}{2r_{10}^7} - \frac{105(\delta_1 - \delta_2)y_0^3(1-\mu)(x_0 - \mu)}{2r_{10}^9} = U_{yx}^0,$$

$$F_{xx}^0 = -\frac{21k\alpha x_0 y_0}{4r_0^{\frac{11}{2}}},$$

$$F_{xy}^0 = k \left(1 + \frac{3\alpha}{2r_0^{\frac{7}{2}}} - \frac{21\alpha y_0^2}{4r_0^{\frac{11}{2}}} \right), \quad (16)$$

$$F_{yx}^0 = -k \left(1 + \frac{3\alpha}{2r_0^{\frac{7}{2}}} - \frac{21\alpha x_0^2}{4r_0^{\frac{11}{2}}} \right),$$

$$F_{xx}^0 = -k = F_{yy}^0, \quad F_{yy}^0 = -F_{xx}^0,$$

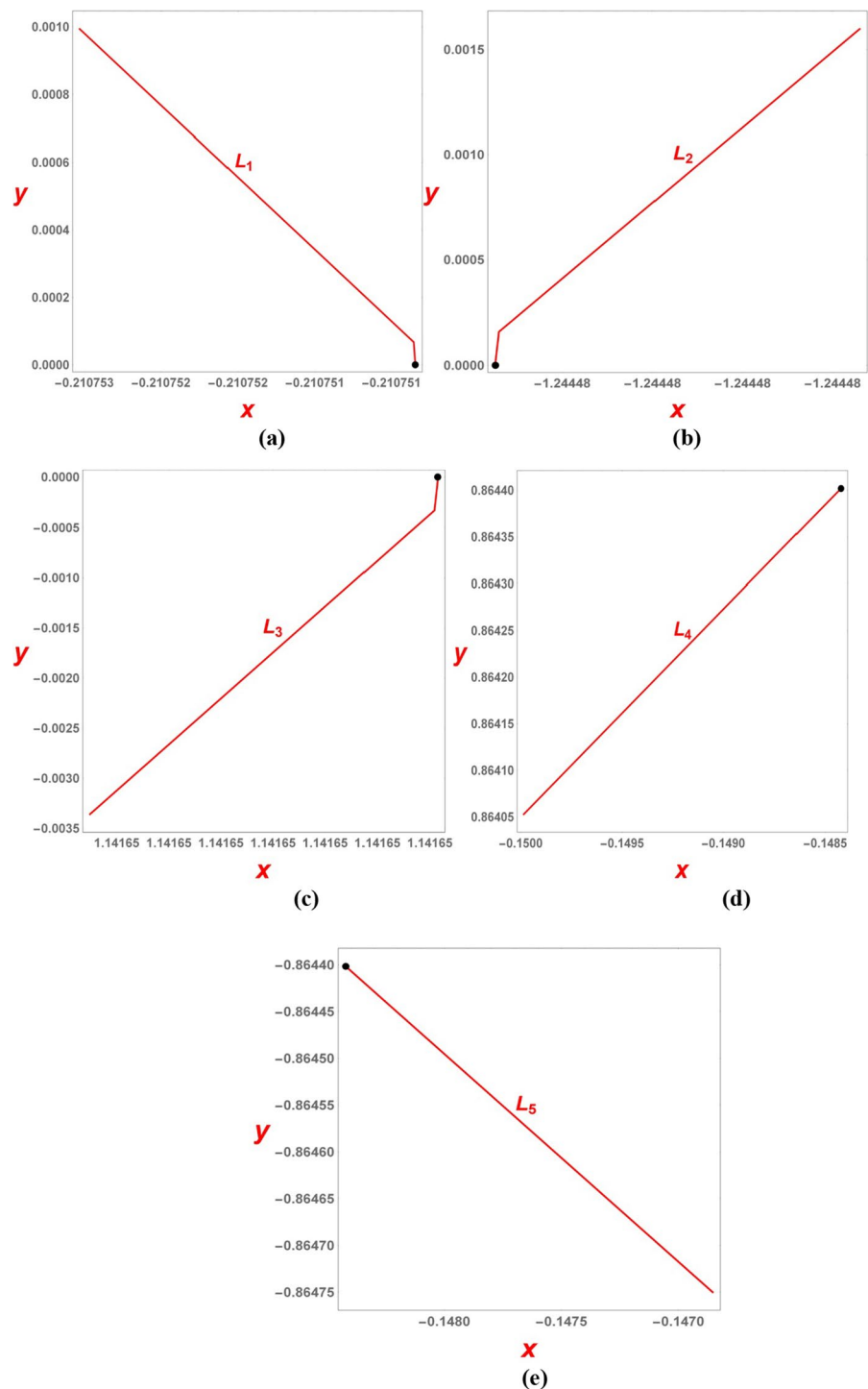
with

$$r_{10} = \sqrt{(x_0 - \mu)^2 + y_0^2}, \quad r_{20} = \sqrt{(x_0 - \mu + 1)^2 + y_0^2}, \quad r_0 = \sqrt{x_0^2 + y_0^2}. \quad (17)$$

Table 2 Effect of the dissipative constant of the SDF of η -Cassiopeiae system on the positions (x_0, y_0) of the five non-collinear equilibrium points for fixed values of $\mu = 0.35088$, $\delta_1 = 0.000125$, $\delta_2 = 0.000118$, $A_2 = 0.0015$ and $\omega = 1.0030$

k	α	L_1	L_2	L_3
0	0	(-0.21075084454, 0)	(-1.24447837228, 0)	(1.14165215088, 0)
10^{-5}	0.01	(-0.2107508446, 1.3765299×10^{-6})	(-1.24447837210, 1.5140041×10^{-5})	(1.14165215031, $-3.1242571 \times 10^{-5}$)
10^{-4}	0.06	(-0.2107508546, 6.7264946×10^{-5})	(-1.24447835283, 1.5664502×10^{-4})	(1.14165208767, $-3.2702600 \times 10^{-4}$)
10^{-3}	0.09	(-0.2107530307, 9.9358974×10^{-4})	(-1.2444763490, 1.5979168×10^{-3})	(1.14164548569, $-3.3578614 \times 10^{-3}$)
k	α	L_4	L_5	
0	0	(-0.14842131731, 0.8644018178080)	(-0.14842131731, -0.8644018178080)	
10^{-5}	0.01	(-0.14843452996, 0.8643988773215)	(-0.14840810438, -0.864404758240)	
10^{-4}	0.06	(-0.14856874851, 0.8643690001809)	(-0.148273851299, -0.8644346284869)	
10^{-3}	0.09	(-0.14998579290, 0.8640528226447)	(-0.146852962912, -0.8647500175898)	

Fig. 3 Locations of the non-collinear equilibrium points, L_i , $i = 1, 2, \dots, 5$ such that L_1 (a), L_2 (b), L_3 (c), L_4 (d) and L_5 (e) of η -Cassiopeiae system for varying Stokes drag force when $\mu = 0.35088$, $\delta_1 = 0.000125$, $\delta_2 = 0.000118$, $A_2 = 0.0015$, and $\omega = 1.0030$ (see Table 2). In each panel, ‘•’ is for $k = 0$ (gravitational case)



4 Numerical application to η -Cassiopeia system

4.1 Analysis of equilibrium points

It is important to stress that the case where $A_2 = k = 0, \omega = 1$ and $\delta_1, \delta_2 \neq 0$ has been studied in detail by Saeed and Zotos [20] where, for $\delta_1, \delta_2 \in [0, 1][0, 1]$, the authors discussed the existence and stability of the equilibrium points. In

Table 3 Effect of the triaxiality parameters of η -Cassiopeiae system on the positions (x_0, y_0) of the five non-collinear equilibrium points for fixed values of $\mu = 0.35088$, $k = 10^{-5}$, $\alpha = 0.05$, $A_2 = 0.0015$ and $\omega = 1.0030$

δ_1	δ_2	L_1	L_2	L_3	$L_{4,5}$
0	0	(−0.210677, 5.66361×10^{-6})	(−1.24453, 1.5564×10^{-5})	(1.14162, -3.2421×10^{-5})	(−0.148386, 0.864438) (−0.148357, −0.864445)
0.000125	0.000118	(−0.210751, 5.65650×10^{-6})	(−1.24448, 1.5560×10^{-5})	(1.14165, -3.2411×10^{-5})	(−0.148436, 0.864399) (−0.148407, −0.864441)
0.000250	0.000236	(−0.210825, 5.64941×10^{-6})	(−1.24443, 1.5555×10^{-5})	(1.14168, -3.2400×10^{-5})	(−0.148486, 0.864359) (−0.148457, −0.86437)
0.000375	0.000354	(−0.210899, 5.64234×10^{-6})	(−1.24438, 1.5550×10^{-5})	(1.14171, -3.2390×10^{-5})	(−0.148536, 0.864320) (−0.148507, −0.86433)

the present work, we have found numerically that three, five, seven, nine or eleven non-collinear EPs of the problem may lie on the plane of motion for varying δ_1 and δ_2 , as depicted in Fig. 2.

In Fig. 2, we set for the η -Cassiopeiae system ($\mu = 0.35088$), $k = 10^{-5}$, $\alpha = 0.05$, $A_2 = 0.01$, and $\omega = 1.01$, because for other values of these parameters the behavior is analogous. The positions of the coordinate of the equilibria on the xy -plane are given by the mutual intersections of the Eq. (8) (blue line) and Eq. (9) (brown line). In these figures, the black dots represent the primaries, while the red dots pinpoint the equilibrium points. Figures 2a—2e show process of the number of non-collinear equilibrium points increasing from three to eleven, for five different cases of δ_1 and δ_2 , (a) $\delta_1 = 0.6, \delta_2 = 0.2$, (b) $\delta_1 = 0.0025, \delta_2 = 0.002$, (c) $\delta_1 = 0.4, \delta_2 = 0.6$, (d) $\delta_1 = 0.2, \delta_2 = 0.5$ and (e) $\delta_1 = 0.3, \delta_2 = 0.65$. As previously noted, the parameters $\mu = 0.35088, k = 10^{-5}, \alpha = 0.05, A_2 = 0.01$, and $\omega = 1.01$ are fixed in all cases. In all cases, the non-collinear equilibrium points are not in symmetric positions with respect to the origin and the xy -plane. It is worth noting that in the absence of Stokes drag effect ($k = 0, \alpha = 0$), the equilibrium point L_4 is symmetric to L_5 , L_6 to L_7 , L_8 to L_9 and L_{10} to L_{11} with respect to the Ox axis.

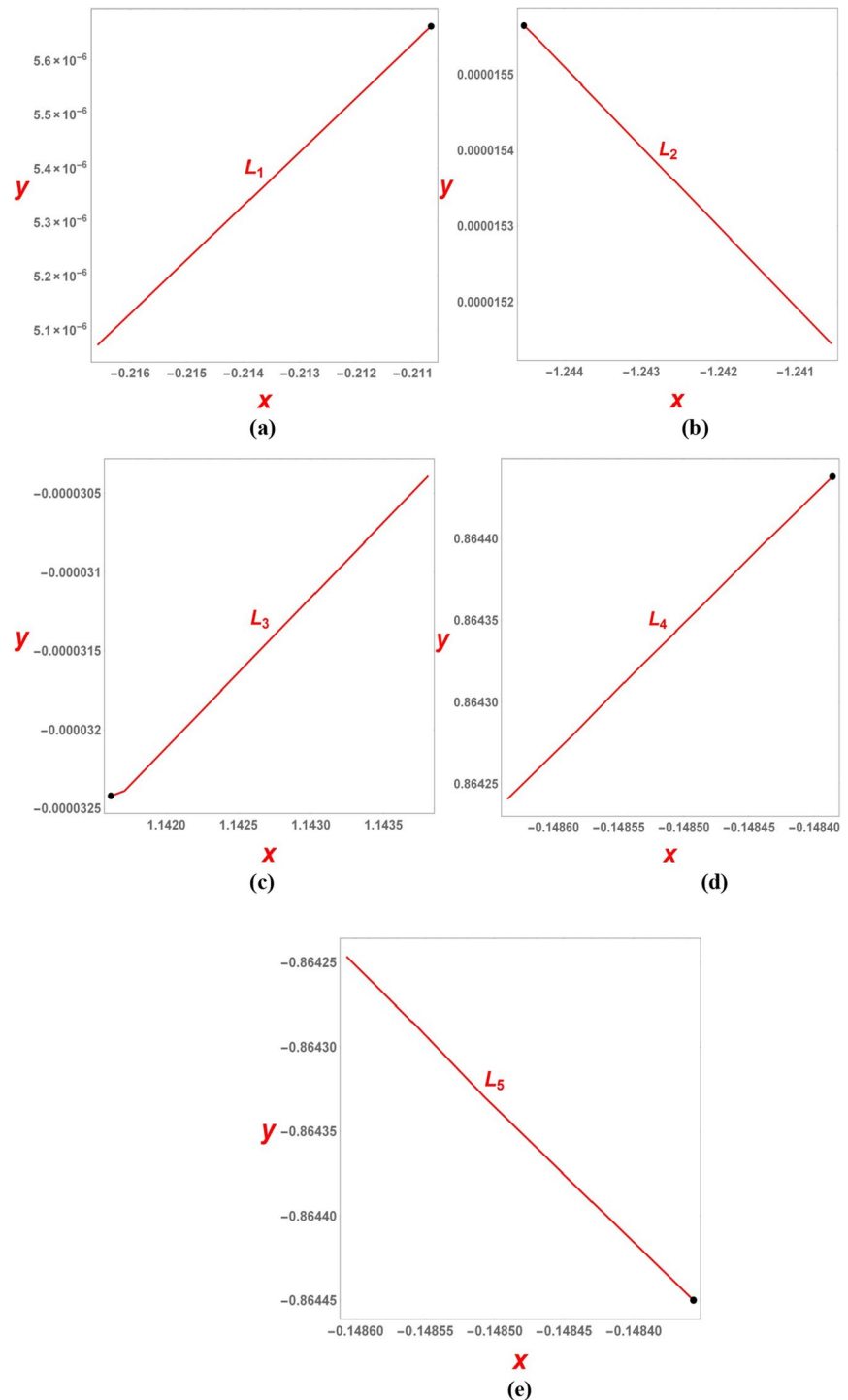
However, we are only concerned here (see, e.g., [19, 25]) with the five non-collinear equilibrium points. In this case, the number of equilibrium points is in agreement with the classical problem (five equilibrium points), and there are no collinear equilibrium points. The points of intersection of these curves display the coordinates (x_0, y_0) of the equilibrium points on the Cartesian plane. As a summary of Fig. 2b, the corresponding locations of the five non-collinear equilibrium points using the specific values of the parameters of the problem stated previously can be identified as follows:

$$L_1(-0.205332, 5.78904 \times 10^{-6}), L_2(-1.24567, 1.51167 \times 10^{-5}), L_3(1.13642, -3.09843 \times 10^{-5}), \\ L_4(-0.144552, 0.858407), L_5(-0.144524, -0.858413)$$

We note that the y -coordinates of $L_{1,2,3}$ are close to zero but not equal to zero. As can be observed from the above results, there are five non-collinear equilibrium points, all of which are not symmetric about the x -axis. These numerical results support our conclusion drawn above. We make mention of the fact that our configuration is comparable to the one in Murray [28] (see Fig. 4 of the paper) for general drag force.

We shall now discuss the positions of the five non-collinear equilibria of the test body as the involved parameters vary. Now, proceeding numerically, we compute the impact that the Stokes drag force has on the positions of the non-collinear equilibrium points for η -Cassiopeiae system. In Table 2, the coordinates of the numerically determined equilibria are shown for fixed values of the chosen model parameters and for various values of the Stokes drag force. We observe that with the increase of the Stokes drag force for fixed values of $\mu = 0.35088, \delta_1 = 0.000125, \delta_2 = 0.000118, A_2 = 0.0015$ and $\omega = 1.0030$, x -coordinates of L_2, L_3, L_5 decrease while at the same time, the y -coordinates increase. Simultaneously, both the x and y -coordinates of L_1 increase whereas the x and y coordinates of L_4 increase and decrease, respectively. However, it is seen that all the corresponding values of $x(L_{1,2,3})$ coordinates are close to each other. This indicates the Stokes drag terms have negligible effect on the x -coordinates of $L_{1,2,3}$. This effect can only be seen in the table. Comparable results can be observed in Fig. 3, panels (a), (b), (c) with overlap in the x -coordinates of the equilibria. We observe that the EPs under the effect of Stokes drag force (and in later cases) have finite but small ordinates for various values of the parameters which are different from that of the classical case. This discussion has been captured in Fig. 3, where the positions of the equilibria have been shown. We observe that with the increase of the Stokes drag of the primaries, the points L_1, L_2, L_3 and L_5 constantly increase their distance from the x -axis, while point L_4 constantly decrease its distance to the x -axis. Furthermore, in the absence of the perturbing force (SDF), we found L_4 and L_5 to be symmetrically

Fig. 4 Locations of the non-collinear equilibrium points, $L_i, i = 1, 2, \dots, 5$ such that L_1 (a), L_2 (b), L_3 (c), L_4 (d) and L_5 (e) of η -Cassiopeiae system for varying triaxiality parameters δ_1 and δ_2 of the bigger primary when $\mu = 0.35088, k = 10^{-5}, \alpha = 0.05, A_2 = 0.0015$ and $\omega = 1.0030$ (see Table 3). In each panel, '•' is for $\delta_1 = \delta_2 = 0$



positioned with respect to the Ox axis. Nevertheless, the presence of the SDF in the structure produces slight asymmetry in the position of such equilibria (see Table 2). All these results are similar with [31].

Similarly, to investigate the influence of the triaxiality parameters (δ_1 and δ_2) of η -Cassiopeiae system on the locations of the equilibrium points, we set for the parameters $\mu = 0.35088, k = 10^{-5}, \alpha = 0.05, A_2 = 0.0015$ and $\omega = 1.0030$. The numerically determined coordinates of these equilibrium points with respect to different values of δ_1 and δ_2 are provided in Table 3. We observe in Table 3 that with the increase of δ_1 in $(0, 0.000375)$ and δ_2 in $(0, 0.000354)$ for fixed values of the chosen model parameters, x -coordinates of L_1, L_3, L_4, L_5 increase while at the same time the y -coordinates decrease;

Table 4 Effect of the oblateness parameter of η –Cassiopeiae system on the positions (x_0, y_0) of the five non-collinear equilibrium points for fixed values of $\mu = 0.35088$, $\delta_1 = 0.000125$, $\delta_2 = 0.000118$, $k = 10^{-5}$, $\alpha = 0.05$ and $\omega = 1.0030$

A_2	L_1	L_2	L_3	$L_{4,5}$
0	(−0.212061, 5.60429 × 10 ^{−6})	(−1.24373, 1.55585 × 10 ^{−5})	(1.14232, −3.25298 × 10 ^{−5})	(−0.149184, 0.864830) (−0.149155, −0.864836)
0.01	(−0.203825, 5.95080 × 10 ^{−6})	(−1.24849, 1.54119 × 10 ^{−5})	(1.13791, −3.17485 × 10 ^{−5})	(−0.144246, 0.861969) (−0.144218, −0.861976)
0.03	(−0.190061, 6.64124 × 10 ^{−6})	(−1.25672, 1.50565 × 10 ^{−5})	(1.12936, −3.02765 × 10 ^{−5})	(−0.134725, 0.856347) (−0.134699, −0.856353)
0.05	(−0.178723, 7.34161 × 10 ^{−6})	(−1.26364, 1.46984 × 10 ^{−5})	(1.12115, −2.89149 × 10 ^{−5})	(−0.125651, 0.850853) (−0.125625, −0.850858)

both the x and y coordinates of L_2 decrease. This effect is shown in Fig. 4. It is seen that the equilibria L_1, L_2, L_3, L_4 and L_5 constantly decrease their distance to the x –axis as the triaxiality parameters increase.

The effect of oblateness of the smaller primary on the positions of the five non-collinear equilibria for the binary system is shown in Table 4 for fixed values of $\mu = 0.35088$, $\delta_1 = 0.000125$, $k = 10^{-5}$, $\alpha = 0.05$, and $\omega = 1.0030$. We observe that with the increase of A_2 from 0 to 0.05 for fixed values of the remaining parameters, x –coordinates of L_1 decrease whereas the y –coordinates increase; x –coordinates of L_2 increase while at the same time the y –coordinates decrease. Simultaneously, both the x and y coordinates of L_3, L_4, L_5 decrease. Figure 5 is a plot of the positions of the equilibria when the oblateness parameter A_2 varies in the interval $[0, 0.1]$ for constant values of the chosen parameters model. We observe that as the oblateness parameter A_2 increases, the points L_2, L_3, L_4, L_5 constantly decrease their distance to the x axis, while the point L_1 moves away from the x axis.

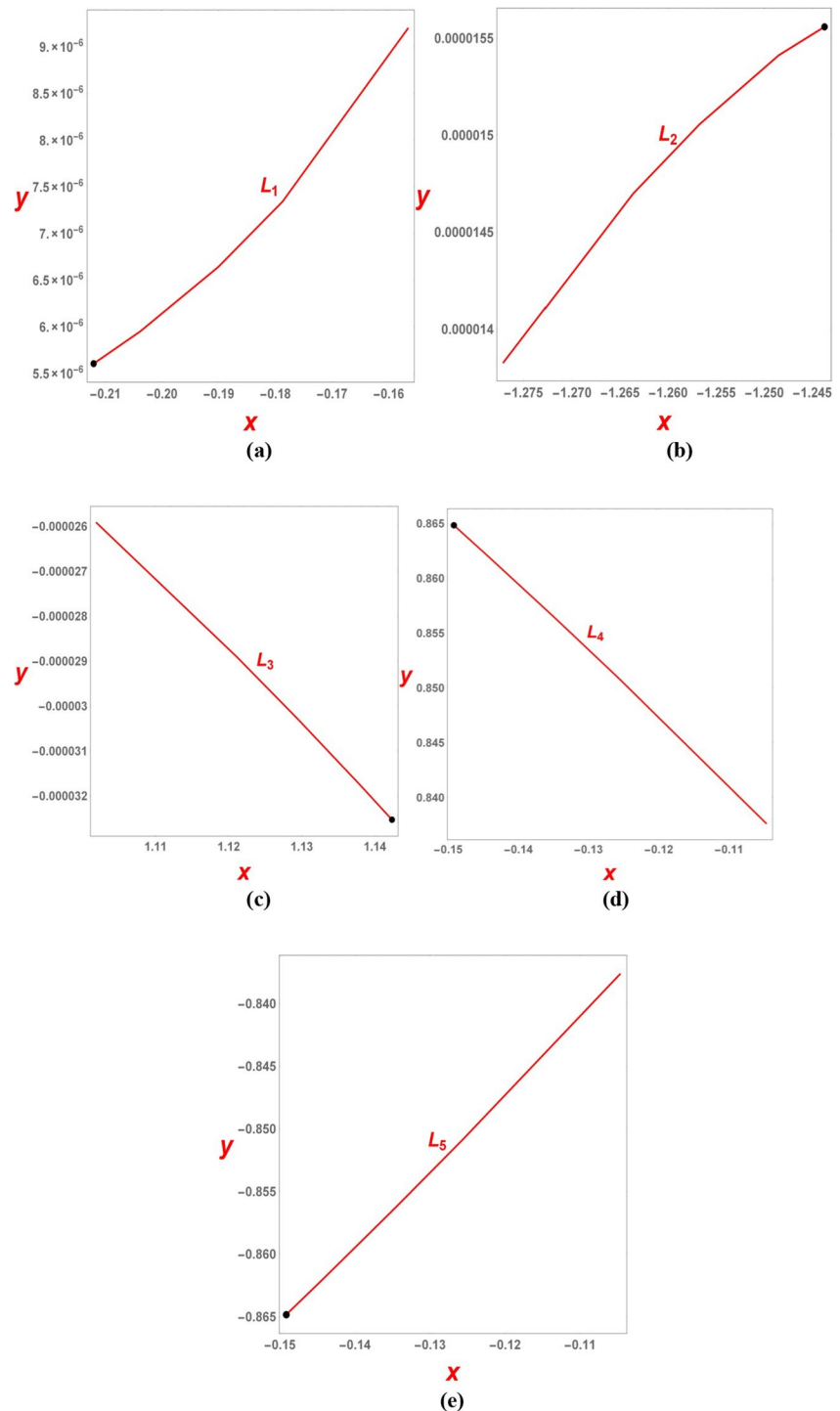
Finally, the effect of small perturbation in centrifugal force of the primaries in the positions of the five non-collinear equilibria for the binary system is shown in Table 5 for fixed values of $\mu = 0.35088$, $\delta_1 = 0.000125$, $k = 10^{-5}$, $\alpha = 0.05$, and $A_2 = 0.0015$. We observe that with the increase of ω from 1 to 1.06 for fixed values of the remaining parameters, x –coordinates of L_1 decrease whereas the y –coordinates increase; x –coordinates of L_5 increase while at the same time the y –coordinates decrease. Simultaneously, both the x and y coordinates of L_2, L_3 and L_4 decrease. Figure 6 is a plot of the positions of the equilibria when the centrifugal force parameter ω varies in the range $[1.0, 1.1]$ for fixed values of the chosen parameters model. We observe that with the increase in the values of ω , point L_1 moves away from the x –axis, while points L_2, L_3, L_4 and L_5 approach the x –axis.

4.2 Stability analysis of the non-collinear equilibrium points

It is interesting to note that the findings for the second partial derivatives as well as the characteristic Eq. (14) coincide with the classical R3BP when $\delta_1 = \delta_2 = A_2 = k = 0$ and $\omega = 1$ (see [2]). It is worth noting that in the case of $\delta_1 = \delta_2 = A_2 = k = 0$, $\omega \neq 1$, the equations of motion are the same as those in [11]. When $A_2 = k = 0$, $\omega = 1$ and $\delta_1, \delta_2 \neq 0$, the equations of motion are the same as those in [20], while the equations of motion correspond with those in [31] when $k_{pr} = 0$ in their potential function and $\nu = \omega = 1$, $A_2 = 0$ in the present study. Furthermore, when $A_2 = 0$, $\nu = \omega = 1$ in the present study, the obtained results of the characteristic equation are in agreement with those in [47] by taking triaxiality of the second primary into consideration, and $q \rightarrow 1$ in their results except for the quantity f_{yx}^0 , which was due to erroneous positive in the coefficient of last term in their Eqs. (15). In general, in the limiting case when $k = 0$ ($\neq 0$), and setting other parameter(s) to zero, we found that the results obtained in this study are in excellent agreement with those obtained previously (see, e.g., [31] and references therein).

In Tables 6, 7, 8, 9, the eigenvalues λ_i , $i = 1, 2, 3, 4$ of each equilibrium point are computed for various values of μ and for assumed constant values of the remaining parameters. Our analysis reveals that for every value of system parameters $k, A_2, \delta_1, \delta_2, \nu$, and ω , the equilibrium points L_1, L_2, L_3 are always linearly unstable since, for the equilibria (L_1, L_2, L_3) , the characteristic Eq. (14) has two real eigenvalues $\lambda_{1,2} = \pm\sigma_1$ (responsible for the equilibria's instability) and the other two are purely imaginary (i.e., $\lambda_{3,4} = \pm i\sigma_2$) or two real eigenvalues $\lambda_1 = -\sigma_1$, $\lambda_2 = \sigma_2$ and a combination of real and imaginary eigenvalues ($\lambda_{3,4} = -\sigma_3 \pm i\sigma_4$), while in contrary, there are values of the problem parameters where the equilibrium points L_4 and L_5 become stable due to pure imaginary roots ($\lambda_{1,2} = \pm i\sigma_1$, $\lambda_{3,4} = \pm i\sigma_2$) or a combination of real and imaginary eigenvalues, i.e., $\lambda_{1,2} = -\sigma_1 \pm i\sigma_2$, $\lambda_{3,4} = -\sigma_3 \pm i\sigma_4$ (stable motion due to the negative real parts of the complex roots). Also, eigenvalues of the form $\lambda_{1,2} = -\sigma_1 \pm i\sigma_2$, $\lambda_{3,4} = \sigma_3 \pm i\sigma_4$ exist at L_4 and L_5 , indicating instability of the equilibria due

Fig. 5 Locations of the five non-collinear equilibrium points, $L_i, i = 1, 2, \dots, 5$ such that L_1 (a), L_2 (b), L_3 (c), L_4 (d) and L_5 (e) of η -Cassiopeiae system for varying oblateness A_2 in the range $[0, 0.1]$ when $\mu = 0.35088$, $\delta_1 = 0.000125$, $\delta_2 = 0.000118$, $k = 10^{-5}$, $\alpha = 0.05$ and $\omega = 1.0030$ (see Table 4). In each panel, '•' is for $A_2 = 0$



to the positive real parts of the complex roots, where $\sigma_i, i = 1, 2, 3, 4$ are real numbers. Based on these results, we may draw the conclusion that the instability of the equilibria (L_4 and L_5) is due to the particular values of the mass parameter of the primary bodies being out of the Routhonian critical mass, i.e., $\mu > \mu_0 = 0.0385208$ (see Table 8). A numerical computation performed on the stability for the binary system η -Cassiopeiae in Table 9 affirms the instability of the equilibria L_4 and L_5

Table 5 Effect of small perturbation in centrifugal force of η –Cassiopeiae system on the positions (x, y) of the five non-collinear equilibrium points for fixed values of $\mu = 0.35088$, $\delta_1 = 0.000125$, $\delta_2 = 0.000118$, $k = 10^{-5}$, $\alpha = 0.05$ and $A_2 = 0.0015$

ω	L_1	L_2	L_3	$L_{4,5}$
1.0	(−0.210788, 5.65141×10^{-6})	(−1.24528, 1.56437×10^{-5})	(1.14257, -3.25775×10^{-5})	(−0.14843583, 0.86554993) (−0.14840678, −0.86555638)
1.02	(−0.210538, 5.68547×10^{-6})	(−1.24001, 1.50965×10^{-5})	(1.13653, -3.14904×10^{-5})	(−0.14843536, 0.85795296) (−0.14840740, −0.85795924)
1.04	(−0.210288, 5.71976×10^{-6})	(−1.23490, 1.14579×10^{-5})	(1.13065, -3.04608×10^{-5})	(−0.14843493, 0.85053565) (−0.14840799, −0.85054178)
1.06	(−0.210039, 5.75428×10^{-6})	(−1.22992, 1.40897×10^{-5})	(1.12492, -2.94844×10^{-5})	(−0.14843452, 0.84329015) (−0.14840855, −0.84329614)

Generally speaking, the equilibria L_1, L_2, L_3 are unstable since there is always a positive real number in the two pairs of roots. Even though unstable, there are still some different cases along with the variation of the parameters. In some instances, a pair of real eigenvalues and a pair of imaginary eigenvalues $\lambda_{1,2} = \pm\sigma_1$, $\lambda_{3,4} = \pm i\sigma_2$ coexist (Table 6) or a combination of real and imaginary eigenvalues $\lambda_1 = -\sigma_1$, $\lambda_2 = \sigma_2$, $\lambda_{3,4} = -\sigma_3 \pm i\sigma_4$ (Tables 7, 8, 9), further enriching the system's dynamical structure.

Similarly, the behavior around L_4 and L_5 varies depending on the system parameters. In some instances, pure imaginary eigenvalues exist (Table 6). However, in certain parameter values, eigenvalues of the form $\lambda_{1,2} = -\sigma_1 \pm i\sigma_2$, $\lambda_{3,4} = -\sigma_3 \pm i\sigma_4$ exist (Table 7), but under certain parameter variations, they exhibit eigenvalues of the form $\lambda_{1,2} = -\sigma_1 \pm i\sigma_2$, $\lambda_{3,4} = \sigma_3 \pm i\sigma_4$ (Tables 8, 9). From the results in Tables 6, 7, 8, 9 we conclude that changes in system parameters significantly affect the topology and stability of the phase space near the equilibrium points.

5 Discussion and conclusion

The location and stability of the equilibrium points (EPs) in the circular restricted three-body problem (CR3BP) under the influence of Stokes drag force (k) with triaxiality of the first primary (δ_1, δ_2) and oblateness of the second primary (A_2) together with small perturbations in the Coriolis and centrifugal forces (v, ω) were studied. We found both analytically and numerically that in the presence of Stokes drag effect the well-known collinear EPs of the CR3BP cease to exist while the respective triangular equilibria do not form equilateral triangles with the primaries. The triaxiality and Stokes drag parameters significantly influence the existence of equilibrium points, leading to situations where either three or five or seven or nine or eleven non-collinear equilibrium points exist in the xy –plane. Using the astrophysical data of the η –Cassiopeiae binary system, the influence of the system's parameters that express perturbing forces were investigated on the positions of the five non-collinear EPs and their linear stability. The locations of these points are affected by the variations in the perturbing forces except small perturbation in the Coriolis force. Specifically, as the Stokes drag force increases, the positions of points L_1, L_2, L_3 and L_5 constantly increase their distance from the x –axis, while, simultaneously, point L_4 constantly decrease its distance to the x –axis. For the triaxiality parameters, we observe that the positions of the equilibria L_1, L_2, L_3, L_4 and L_5 constantly decrease their distance to the x –axis as the triaxiality coefficients increase. Also, we observe that as the oblateness parameter increases, the positions of the points L_2, L_3, L_4, L_5 constantly decrease their distance to the x axis, while, at the same time, point L_1 moves away from the x axis. Furthermore, as ω increases, the equilibrium point L_1 moves away from the x –axis, while, simultaneously, the positions of points L_2, L_3, L_4 and L_5 approach the x –axis.

We also assessed the linear stability of these equilibrium points across a wide range of system's parameter values and determined the nature of motion of the infinitesimal mass. We found that the equilibria L_1, L_2, L_3 are generally unstable since there is always a positive real number in the two pairs of roots in contrast to the equilibrium L_4 and L_5 which can be stable for certain parameter values. Additionally, our results suggest that the stability of equilibrium points is sensitive to variations in system parameters, resulting in different topological structure near the equilibrium points. This finding reinforces the conclusions of [8, 14, 31, 46], and references therein regarding the circular R3BP under the influence of the aforementioned perturbations.

Fig. 6 Locations of the five non-collinear equilibrium points $L_i, i = 1, 2, \dots, 5$ such that L_1 (a), L_2 (b), L_3 (c), L_4 (d) and L_5 (e) of η -Cassiopeiae system for varying centrifugal force ω in the range $[1.0, 1.1]$ when $\mu = 0.35088$, $\delta_1 = 0.000125$, $\delta_2 = 0.000118$, $k = 10^{-5}$, $\alpha = 0.05$ and $A_2 = 0.0015$ (see Table 5). In each panel, '•' is for $\omega = 1$

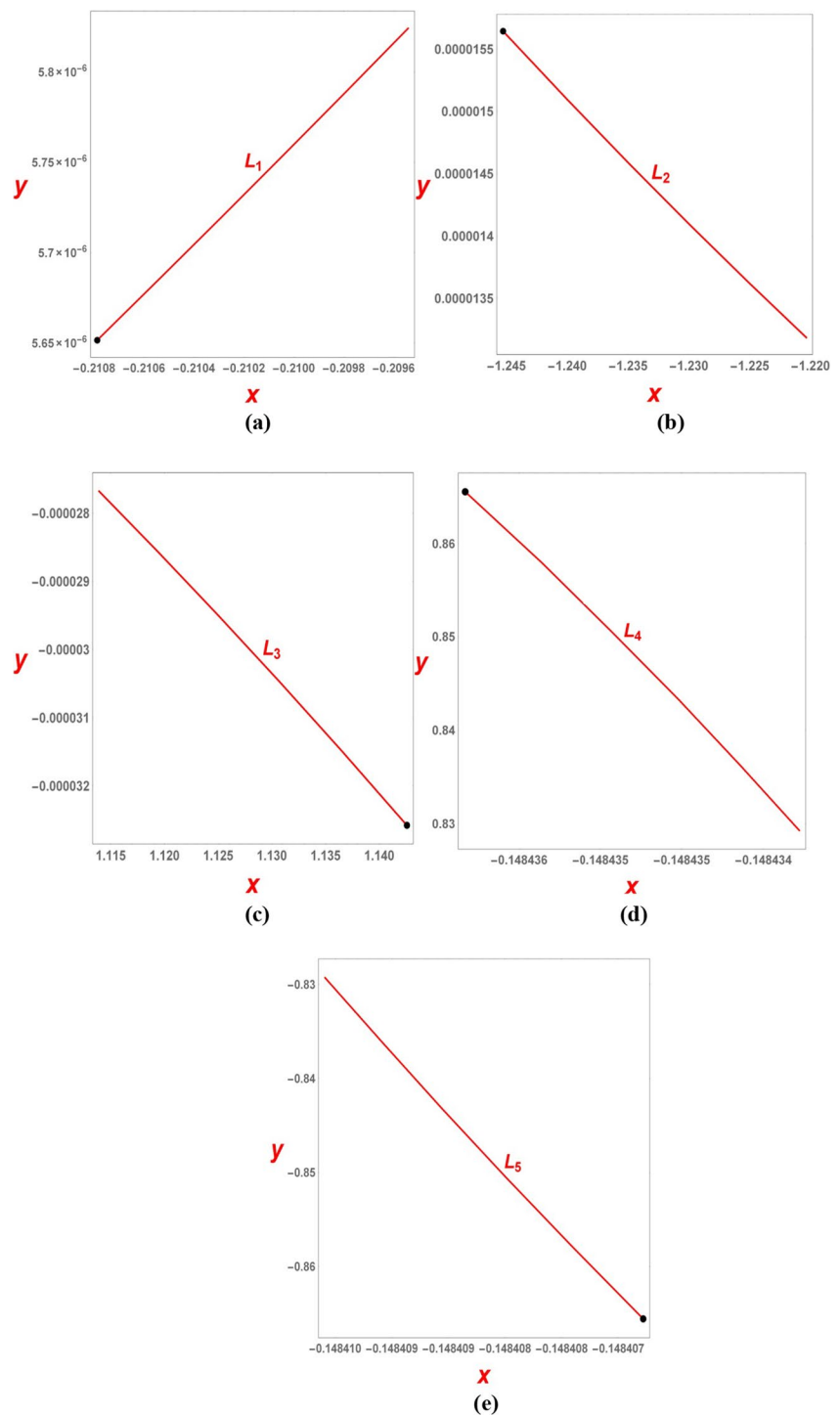


Table 6 Locations (x_0, y_0) and stability of the non-collinear EPs in the absence of the SDF ($k = 0$) when $\mu = 0.02$, $\delta_1 = 0.000130$, $\delta_2 = 0.000125$, $A_2 = 0.0002$, $\omega = 1.02$ and $\nu = 1.01$

L_i	(x_0, y_0)	$\lambda_{1,2}$	$\lambda_{3,4}$
L_1	$(-0.801588, 0)$	± 6.27847	$\pm 2.37172 i$
L_2	$(-1.177300, 0)$	± 2.16019	$\pm 1.86344 i$
L_3	$(1.001680, 0)$	± 0.23104	$\pm 1.02747 i$
$L_{4(5)}$	$(-0.479838, \pm 0.85839)$	$\pm 0.402391 i$	$\pm 0.926607 i$

Table 7 Positions (x_0, y_0) and stability of the five non-collinear equilibrium points under the combined effect of perturbations $\delta_1 = 0.000130, \delta_2 = 0.000125, k = 10^{-5}, \alpha = 0.05, A_2 = 0.0002, \omega = 1.02$ and $\nu = 1.01$ when $\mu = 0.02$

L_i	(x_0, y_0)	$\lambda_{1,2}$	$\lambda_{3,4}$
L_1	$(-0.80159, 2.16629 \times 10^{-6})$	$-2.9956, 2.99559$	$-5.18302 \times 10^{-6} \pm 2.37172i$
L_2	$(-1.17730, 5.64324 \times 10^{-6})$	$-2.1602, 2.16019$	$-5.10433 \times 10^{-6} \pm 1.86344i$
L_3	$(1.001680, -5.91763 \times 10^{-4})$	$-0.231037, 0.231029$	$-6.20556 \times 10^{-6} \pm 1.02747i$
L_4	$(-0.480038, 0.858278)$	$-2.97212 \times 10^{-6} \pm 0.402541i$	$-7.02788 \times 10^{-6} \pm 0.926539i$
L_5	$(-0.479638, -0.858507)$	$-2.97354 \times 10^{-6} \pm 0.402254i$	$-7.02646 \times 10^{-6} \pm 0.926671i$

Table 8 Locations (x_0, y_0) and stability of the five non-collinear equilibrium points under the combined effect of perturbations $\delta_1 = 0.00015, \delta_2 = 0.00011, k = 10^{-4}, \alpha = 0.1, A_2 = 0.00035, \omega = 1.03$ and $\nu = 1.02$ when $\mu = 0.05$

L_i	(x_0, y_0)	$\lambda_{1,2}$	$\lambda_{3,4}$
L_1	$(-0.712935, 2.16922 \times 10^{-5})$	$-3.18599, 3.18589$	$-5.48284 \times 10^{-5} \pm 2.5007i$
L_2	$(-1.223180, 6.91665 \times 10^{-5})$	$-2.02077, 2.16019$	$-5.21231 \times 10^{-5} \pm 1.79492i$
L_3	$(1.010940, -2.46813 \times 10^{-3})$	$-0.359717, 0.359658$	$-7.04579 \times 10^{-5} \pm 1.07452i$
L_4	$(-0.450337, 0.854253)$	$-0.161637 \pm 0.74981i$	$0.161537 \pm 0.749685i$
L_5	$(-0.448632, -0.855195)$	$-0.159302 \pm 0.749352i$	$0.159202 \pm 0.749226i$

Table 9 Locations (x_0, y_0) and stability of the five non-collinear equilibrium points for η -Cassiopeiae system ($\mu = 0.35088$) when $\delta_1 = 0.000130, \delta_2 = 0.000125, A_2 = 0.0002, \omega = 1.02, k = 10^{-5}, \alpha = 0.05$ and $\nu = 1.01$

L_i	(x_0, y_0)	$\lambda_{1,2}$	$\lambda_{3,4}$
L_1	$(-0.211671, 5.64002 \times 10^{-6})$	$-3.74006, 3.74007$	$-1.67673 \times 10^{-5} \pm 2.85748i$
L_2	$(-1.239350, 1.51174 \times 10^{-5})$	$-1.38845, 1.38844$	$-5.18473 \times 10^{-6} \pm 1.44806i$
L_3	$(1.1371, -3.15917 \times 10^{-5})$	$-0.959532, 0.959522$	$-5.37519 \times 10^{-6} \pm 1.25044i$
L_4	$(-0.14909, 0.858321)$	$-0.61684 \pm 0.943774i$	$0.61683 \pm 0.943772i$
L_5	$(-0.149062, -0.858327)$	$-0.616832 \pm 0.943769i$	$0.616822 \pm 0.943767i$

Also, some remarkable articles should be cited, which concern the problem under consideration, [48, 49]

Finally, the numerical and graphics of the paper were produced, using the latest version of Mathematica [50]

Acknowledgements We would like to sincerely thank the anonymous reviewers for their insightful and helpful comments on our manuscript.

Author contributions A.E.V and J.M.G were both involved in the Conceptualization, methodology, formal analysis, writing—original draft preparation, writing—review and editing and preparations of the figures and tables. All authors have read and agreed to the published version of the manuscript.

Funding This research received no external funding.

Data availability The original contributions presented in the study are included in the article/supplementary Material, further inquiries can be directed to the corresponding author.

Declarations

Ethics approval and consent to participate Not applicable.

Consent for publication Not applicable.

Competing Interests The authors declare no competing interests.

Open Access This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License, which permits any non-commercial use, sharing, distribution and reproduction in any medium or format, as long as you give appropriate credit to

the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if you modified the licensed material. You do not have permission under this licence to share adapted material derived from this article or parts of it. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by-nc-nd/4.0/>.

References

- Koon, W. S., Lo, M. W., Marsden, J. E., Ross, S. D. Dynamical Systems, The Three-Body Problem, and Space Mission Design. (2011). http://www.cds.caltech.edu/~marsden/books/Mission_Design.html
- Szebehely V. Theory of Orbits. In: The restricted problem of three bodies. New York: Academic Press; 1967.
- Singh J, Perdiou AE, Gyegwe JM, Kalantonis VS. Periodic orbits around the collinear equilibrium points for binary Sirius, Procyon, Luhman 16, (-Centuari and Luyten 726–8 systems: the spatial case. J Phys Commun. 2017. <https://doi.org/10.1088/2399-6528/aa8976>.
- Vincent EA, Perdiou AE, Singh J. Out-of-plane equilibrium equilibrium points in the r3bp with radiation pressure, Poynting-Robertson drag and angular velocity variation. In: Rassias TM, Pardalos P, editors. Mathematical analysis, differential equations and applications. London: World scientific; 2024.
- Wintner A. The analytical foundations of celestial mechanics. New Jersey: Princeton University Press; 1947.
- Hou XY, Xin XS, Feng JL. Forced motions around triangular libration points by solar radiation pressure in a binary asteroid system. Astrodynamics. 2020;4:17–20. <https://doi.org/10.1007/s42064-019-0060-2>.
- Čuk M, Burns J. Effects of thermal radiation on the dynamics of binary NEAs. Icarus. 2005;176:418–31. <https://doi.org/10.1016/j.icarus.2005.02.001>.
- Vincent AE, Perdiou AE, Perdios EA. Existence and stability of equilibrium points in the R3BP with triaxial-radiating primaries and an oblate massless body under the effect of the circumbinary disc. Fron Astron Space Sci. 2022;9: 877459. <https://doi.org/10.3389/fspas.2022.877459>.
- Szebehely V. Stability of the points of equilibrium in the restricted problem. Astron J. 1967;72:7–9. <https://doi.org/10.1086/110195>.
- Subbarao PV, Sharma KR. A note on the stability of the triangular points of equilibrium in the restricted three-body problem. Astron Astrophys. 1975;43:381–3.
- Bhatnagar KP, Hallan PP. Effect of perturbations in coriolis and centrifugal forces on the stability of libration points in the restricted problem. Celest Mech Dyn Astron. 1978;18:105–12. <https://doi.org/10.1007/BF01228710>.
- Abdulraheem A, Singh J. Combined effects of perturbations, radiation and oblateness on the stability of equilibrium points in the restricted three-body problem. Astron J. 2006;131:1880–5. <https://doi.org/10.1086/499300/pdf>.
- Sharma RK, Taqvi ZA, Bhatnagar KB. Existence and stability of libration points in the restricted three-body problem when the primaries are triaxial rigid bodies. Celest Mech Dyn Astron. 2001;79:119–33. <https://doi.org/10.1023/A:1011168605411>.
- Singh J, Begha JM. Stability of equilibrium points in the generalized perturbed restricted three-body problem. Astrophys Space Sci. 2011;331:511–9. <https://doi.org/10.1007/s10509-010-0464-1>.
- Singh J, Ahmad MS. Effects of perturbations on the stability of equilibrium points in the CR3BP with luminous and heterogeneous spheroid primaries. Sci Rep. 2022;12:2819. <https://doi.org/10.1038/s41598-022-06328-y>.
- Aboulmagd El, Guirao J. On the perturbed restricted three-body problem. Appl Math Nonlinear Sci. 2016;1:123–44. <https://doi.org/10.2104/AMNS.2016.1.00010>.
- Kalantonis VS, Douskos CN, Perdios EA. Numerical determination of homoclinic and heteroclinic orbits at collinear equilibria in the restricted three-body problem with oblateness. Celest Mech Dyn Astron. 2006;94:135–53. <https://doi.org/10.1007/s10569-005-4441-x>.
- Singh J, Perdiou A, Gyegwe JM, Perdios EA. Periodic solutions around the collinear equilibrium points in the perturbed restricted three-body problem with triaxial and radiating primaries for binary HD 191408, Kruger 60 and HD 155876 systems. Appl Math Comput. 2018;325:358–74. <https://doi.org/10.1016/j.amc.2017.11.052>.
- Elshaboury SM, Aboulmagd El, Kalantonis VS, Perdios EA. The planar restricted three-body problem when both primaries are triaxial rigid bodies: Equilibrium points and periodic orbits. Astrophys Space Sci. 2016;361:315. <https://doi.org/10.1007/s10509-016-2894-x>.
- Saeed T, Zotos EE. On the equilibria of the restricted three-body problem with a triaxial rigid body - I oblate primary. Result Phys. 2021;23: 103990. <https://doi.org/10.1016/j.rinp.2021.103990>.
- Alrebdil HI, Dubeibe FL, Zotos EE. On the equilibria of the restricted three-body problem with a triaxial rigid body II: prolate primary. Result Phys. 2022;53: 105623. <https://doi.org/10.1016/j.rinp.2022.105623>.
- Oberti P, Vienne A. An upgraded theory for Helene, Telesto, and Calypso. Astron Astrophys. 2003;397:353–9. <https://doi.org/10.1051/0004-6361:20021518>.
- Jancart S, Lemaître A. Dissipative forces and external resonances. Celest Mech Dyn Astr. 2001;81:75–80. <https://doi.org/10.1023/A:1013311204539>.
- Murray CD, Dermott SF. Solar system dynamics. Cambridge: Cambridge University Press; 1999.
- Celletti A, Stefanelli L, Lega E, Froeschlé C. Some Results on the global dynamics of the regularized restricted three-body problem with dissipation. Celest Mech Dyn Astron. 2011;109:265–84. <https://doi.org/10.1007/s10569-010-9326-y>.
- Idrisi MJ, Jain M. Restricted three-body problem with Stokes drag effect when the less massive primary is an ellipsoid. Int J Adv Astron. 2016. <https://doi.org/10.1441/ijaa.v4il.6140>.
- Beauge C, Ferraz-Mello S. Resonance trapping in the primordial solar nebula: the case of a stokes drag dissipation. Icarus. 1993;103:301–18. <https://doi.org/10.1006/icar.1993.1072>.
- Murray CD. Dynamical effects of drag in the circular restricted three body problems: 1 Location and stability of the Lagrangian equilibrium points. Icarus. 1994;112:465–184. <https://doi.org/10.1006/icar.1994.1198>.

29. Jain M, Aggarwal R. Restricted three body problem with stokes drag effect. *IJAA*. 2015;5:95–105. <https://doi.org/10.4236/ijaa.2015.52013>.
30. Jain M, Aggarwal RA. Study of non-collinear libration points in restricted three body problem with Stokes drag effect when smaller primary is an oblate spheroid. *Astrophys Space Sci*. 2015;358:51. <https://doi.org/10.1007/s10509-015-2457-6>.
31. Vincent AE, Perdiou AE. Existence and stability of equilibrium points under the influence of Poynting-Robertson and Stokes drags in the restricted three-body problem. In: Rassias THM, Parasidis IN, Providas E, editors. *Mathematical Analysis in Interdisciplinary Research*. Cham: Springer; 2021.
32. Ragos O, Zafropoulos FA. A numerical study of the influence of the Poynting-Robertson effect on the equilibrium points of the photo-gravitational restricted three– body problem I Coplanar case. *Astron Astrophys*. 1995;300:568–78.
33. Liou JC, Zook HA, Jackson AA. Radiation pressure, Poynting-Robertson drag, and solar wind drag in the restricted three-body problem. *Icarus*. 1995;116:186–201. <https://doi.org/10.1006/icar.1995.1120>.
34. Vincent AE, Kalantonis VS. Motion around the equilibrium points in the photogravitational R3BP under the effects of Poynting-Robertson drag, circumbinary belt and triaxial primaries with an oblate infinitesimal body: application on achird binary system. In: Rassias ThM, Pardalos P, editors. *Analysis, geometry, nonlinear optimization and applications*. London: World Scientific; 2023.
35. Vincent AE, Perdiou AE. Poynting-Robertson and oblateness effects on the equilibrium points of the perturbed R3BP: application on Cen X-4 binary system. In: Rassias TM, editor. *Nonlinear analysis, differential equations, and applications*. Cham: Springer; 2021.
36. Arif M, Sagar RK. Study of the restricted three body problem when one primary is a uniform circular disk. *Appl Appl Math*. 2018;13:160–72.
37. Gao F, Wang R. Bifurcation analysis and periodic solutions of the HD 191408 system with triaxial and radiative perturbations. *Universe*. 2020;6:35. <https://doi.org/10.3390/universe6020035>.
38. Abouelmagd EI, Pal AK, Guirao JLG. Analysis of nominal halo orbits in the Sun– Earth system. *Arch Appl Mech*. 2021;91(12):4751–63. <https://doi.org/10.1007/s00419-021-02031-y>.
39. Ershkov S, Leshchenko D, Rachinskaya A. Note on the trapped motion in ER3BP at the vicinity of barycenter. *Arch Appl Mech*. 2021;91(3):997–1005. <https://doi.org/10.1007/s00419-020-01801-4>.
40. Singh J. The equilibrium points in the perturbed R3BP with triaxial and luminous primaries. *Astrophys Space Sci*. 2013;346:41–50. <https://doi.org/10.1007/s10509-013-1420-7>.
41. Johnston's archive. Available online: <https://www.johnstonsarchive.net/astro/nearstar.html>. Accessed 17 June 2022
42. Mia R. Lie-series solution of restricted three-body problem: application to binary stellar systems. *J Astronaut Sci*. 2019;67:59–76. <https://doi.org/10.1007/s40295-019-00172-5>.
43. Mia R, Kushvah BS. Stability and fourier-series periodic solution in the binary stellar systems. *Few-Body Syst*. 2016. <https://doi.org/10.1007/s00601-016-1112-2>.
44. Mia R. Dynamics of infinitesimal particle in the framework of photo-gravitational three-body problem. *J Appl Nonlinear Dyn*. 2021;10:173–86. <https://doi.org/10.5890/JAND.2021.03.011>.
45. Mia R, Nayak SP, Tiwary RD. Equilibrium points and their linear stability analysis under the effect of dark matter halo. *Ind J Phys*. 2024;8:2665–76. <https://doi.org/10.1007/s12648-024-03097-4>.
46. Elipe A. Wrong hypotheses in the generalized RTBP. *Astrophys Space Sci*. 2024;369:22. <https://doi.org/10.1007/s10509-024-04286-7>.
47. Idrisi MJ, Ullah MS. A study of non-collinear libration points under the combined effects of radiation pressure and stokes drag. *New Astron*. 2021;82: 101441. <https://doi.org/10.1016/j.newast.2020.101441>.
48. Singh J, Kalantonis VS, Gyegwe JM, Perdiou AE. Periodic motions around the collinear equilibrium points of the r3bp where the primary is a triaxial rigid body and the secondary is an oblate spheroid. *Astrophys J Suppl Ser*. 2016;227:13. <https://doi.org/10.3847/0067-0049/227/2/13>.
49. Agbogidi BR, Vincent AE, Afere BAE, Ibrahim SD. Equilibrium points in the perturbed rotating mass dipole system with oblateness. *New Astron*. 2025;116: 102344. <https://doi.org/10.1016/j.newast.2024.102344>.
50. Wolfram S. *The mathematica book*. Wolfram Research: Inc; 2003.