



Solutions of Schrodinger equation for the modified Mobius square plus Kratzer potential

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Abstract In this paper, we obtain the approximate solutions of the Schrodinger equation with the modified Mobius square plus Kratzer potential using the Nikiforov–Uvarov method and employing the approximation scheme for the centrifugal term. We obtain the energy eigenvalue equation and corresponding wave functions. Finally, some numerical results and special cases are also reported.

1 Introduction

In theoretical physics and other physics-related fields, much research interest has been directed toward obtaining and discussing the exact solutions of Schrodinger wave equation for different potentials, during the past years [1–5]. The reason is because the solutions contain sufficient information to explain the quantum system under probe. However, reports have shown that exact solutions of Schrodinger equation are only possible in few cases [6–8]. So far, various techniques have been applied for the purpose of obtaining the exact and approximate solutions of the Schrodinger equation for different potentials. These techniques include the Nikiforov–Uvarov (NU) method [9–12], supersymmetric quantum mechanics [13], the asymptotic iteration method (AIM) [14], series expansion [15], algebraic approach [16] and others.

The Kratzer potential [17] is an important diatomic potential in physics. It is a useful model in describing molecular structure interactions in quantum mechanics [18]. The energy spectrum of diatomic CO molecule with different quantum numbers can be successfully accounted for applying the Kratzer potential. Recently, Bayrak et al. [19] studied the exact analytical solution of the Schrodinger equation in the presence of Kratzer potential by using the asymptotic iteration method. The energy eigenvalues and corresponding wave functions were obtained for some diatomic molecules. The solution of Schrodinger equation with noncentral modified Kratzer potential has been obtained using the factorization method [20]. Setare and Karimi [16] used the algebraic approach to obtain the exact bound state energy eigenvalues and the corresponding eigenfunctions of a Schrodinger equation in the presence

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of Kratzer potential, and with the energy levels appearing nonequidistant. Berkdemir et al. [21] obtained the solutions of the Schrodinger equation for the modified central Kratzer potential by means of the Nikiforov–Uvarov method.

The modified Mobius square potential is known to be a more general case of Morse and Hulthen potentials [22]. Recently, Ikot et al. [23] studied symmetry limits of $(D + 1)$ -dimensional Dirac equation with the Mobius square potential. They obtained the energy eigenvalues and corresponding eigenfunctions using the Nikiforov–Uvarov method. By making use of factorization method, Okorie et al. [24] obtained the bound state solutions of Schrodinger equation with modified Mobius potential and applied their result to study thermodynamic properties. Antia et al. [25] solved the Klein–Gordon equation with equal scalar and vector Mobius square plus Yukawa potentials in D -dimensions using the parametric form of Nikiforov–Uvarov method.

Thus, it is important to investigate the solutions of the Schrodinger equation for the modified Mobius square potential plus Kratzer potential. The modified Mobius square potential plus Kratzer potential considered in the present work is of the form:

$$V(r) = -V_0 \left(\frac{A + Be^{-2\alpha r}}{1 - e^{-2\alpha r}} \right)^2 + \frac{V_1 e^{-2\alpha r}}{1 - e^{-2\alpha r}}, \quad (1)$$

where V_0 , A , B , V_1 and α are the potential depth, the range of the potential, molecular bond length, coupling parameter and the screening parameter, respectively. When $V_0 = 0$, the potential reduces to Kratzer potential, and with $V_1 = 0$, Eq. (1) changes to the Mobius potential. In the present work, the Nikiforov–Uvarov method is used to solve the Schrodinger equation for the potential introduced in Eq. (1). The paper is structured as follows: A brief review of Nikiforov–Uvarov method is discussed in Sect. 2. In Sect. 3, we present the solution to radial part of Schrodinger equation. In Sect. 4, discussions of numerical results and special cases of this model are considered. Finally, a brief conclusion is given in Sect. 5.

2 Nikiforov–Uvarov (NU) method

The NU method reduces linear second-order differential equations to a generalized equation in its hypergeometric form. This method has been successfully used to solve many differential equations of quantum mechanics including the Schrodinger, relativistic Dirac, Klein–Gordon, Duffin–Kemmer–Petiau (DKP) and spinless-Salpeter equations. The NU method is noted to be tedious and time-consuming method, yet it has received numerous attentions in the course of bound state. To popularize it more, Tezcan and Sever [26] derived a parametric form of the NU method from the conventional NU method. The parametric NU method is a straightforward method which is simpler and accurate for the determination of energy equation and the corresponding wave function. To apply the parametric NU method, a general form of a differential equation is given as [26,27]

$$\frac{d^2\psi(s)}{ds^2} + \frac{\alpha_1 - \alpha_2 s}{s(1 - \alpha_3 s)} \frac{d\psi(s)}{ds} + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{s^2(1 - \alpha_3 s)^2} \psi(s) = 0 \quad (2)$$

The conditions for energy equation and the wave functions are, respectively, given as

$$\begin{aligned} (\alpha_2 - \alpha_3)n + \alpha_3 n^2 - (2n + 1)\alpha_5 + (2n + 1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \\ + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0, \end{aligned} \quad (3)$$

$$\psi(s) = N_n s^{\alpha_{12}} (1 - \alpha_3 s)^{-\alpha_{12} - (\alpha_{13}/\alpha_3)} P_n^{(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10}-1)} (1 - 2\alpha_3 s) \quad (4)$$

where

$$\left. \begin{aligned} \alpha_4 &= \frac{1}{2}(1 - \alpha_1), \quad \alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3), \quad \alpha_6 = \alpha_5^2 + \xi_1, \quad \alpha_7 = 2\alpha_4\alpha_5 - \xi_2, \\ \alpha_8 &= \alpha_4^2 + \xi_3, \quad \alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6, \quad \alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, \\ \alpha_{11} &= \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}), \quad \alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \quad \alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \end{aligned} \right\}, \quad (5)$$

and P_n is the orthogonal Jacobi polynomial.

3 Schrodinger equation with modified Mobius square plus Kratzer potential

Given a Schrodinger equation in spherical coordinate [28]

$$-\frac{\hbar^2}{2\mu} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi(r, \theta, \phi) = E \psi(r, \theta, \phi) \quad (6)$$

where E is the energy, $\hbar$ is the reduced Planck constant, μ is the reduced mass, and ψ is the wave function of particle. Defining $\psi(r, \theta, \phi) = \frac{R(r)}{r} Y_{lm}(\theta, \phi)$, the radial part of the Schrodinger equation with a centrifugal term is given by

$$\frac{d^2 R_{nl}(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2} (E - V(r)) - \frac{l(l+1)}{r^2} \right] R_{nl}(r) = 0, \quad (7)$$

where n and l are the radial and orbital angular momentum quantum numbers, respectively.

Substituting Eq. (1) into (7) gives

$$\frac{d^2 R_{nl}}{dr^2} + \left[\frac{2\mu}{\hbar^2} \left(E + V_0 \left(\frac{A + Be^{-2ar}}{1 - e^{-2ar}} \right)^2 - \frac{V_1 e^{-2ar}}{1 - e^{-2ar}} \right) - \frac{l(l+1)}{r^2} \right] R_{nl}(r) = 0. \quad (8)$$

It is clear that Eq. (8) cannot be analytically solvable with $l \neq 0$, due to the centrifugal term. However, to address this constrain, we apply Green–Aldrich approximation Scheme [29] to deal with the centrifugal term.

$$\frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2ar}}{(1 - e^{-2ar})^2}, \quad (9)$$

The centrifugal term and its approximation are shown in Fig. 1. To solve the differential equation of Eq. (8) using parametric Nikiforov–Uvarov method after the substitution of Eq. (9), we make a simple transformation of the form $s = e^{-2ar}$, and this turns Eq. (8) to the form

$$\frac{d^2 R(r)}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dR(s)}{ds} + \frac{-\varepsilon^2 s^2 + \varphi s - \phi}{s^2(1-s)^2} R(r) = 0, \quad (10)$$

where

$$\varepsilon^2 = -\frac{\mu(E + V_1 + V_0 B^2)}{2\alpha^2 \hbar^2}, \quad (11)$$

$$\varphi = \frac{\mu(2ABV_0 - 2E - V_1)}{2\alpha^2 \hbar^2} - l(l+1), \quad (12)$$

$$\phi = -\frac{\mu(E + V_0 A^2)}{2\alpha^2 \hbar^2}. \quad (13)$$

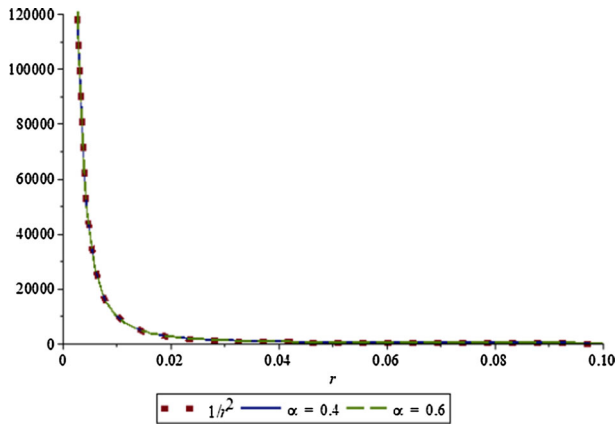


Fig. 1 A plot of the variation in the centrifugal term $1/r^2$ and the approximation as a function of r for various values of $\alpha = 0.4$ and $\alpha = 0.6$

Comparing Eq. (2) with (10), we deduce the values of the parametric constants in Eq. (5) as follows

$$\left. \begin{aligned} \alpha_1 = \alpha_2 = \alpha_3 = 1, \quad \alpha_4 = 0, \quad \alpha_5 = -\frac{1}{2}, \quad \alpha_6 = \varepsilon^2 + \frac{1}{4}, \quad \alpha_7 = -\varphi, \quad \alpha_8 = \phi, \\ \alpha_9 = \frac{1}{2} \sqrt{(1+2l)^2 - \frac{2\mu V_0(A+B)^2}{\alpha^2 \hbar^2}}, \quad \alpha_{10} = 1 + 2\sqrt{\phi}, \\ \alpha_{11} = 2 \left(1 + \sqrt{\phi} \right) + \sqrt{(1+2l)^2 - \frac{2\mu V_0(A+B)^2}{\alpha^2 \hbar^2}}, \quad \alpha_{12} = \sqrt{\phi}, \\ \alpha_{13} = -\frac{1}{2} \left(1 + \sqrt{(1+2l)^2 - \frac{2\mu V_0(A+B)^2}{\alpha^2 \hbar^2}} \right) - \sqrt{\phi} \end{aligned} \right\}. \quad (14)$$

Substituting the values of the parametric constants in Eq. (5) into Eqs. (3) and (4), respectively, we have the energy equation and the corresponding unnormalized radial wave function as

$$E = -AV_0 - \frac{2\alpha^2 \hbar^2}{\mu} \left(\frac{\frac{\mu(V_1 - 2ABV_0 - V_0A^2)}{2\alpha^2 \hbar^2} + \frac{1}{2} + n(n+1) + \ell(\ell+1) + \left(n + \frac{1}{2}\right) \sqrt{(1+2l)^2 - \frac{2\mu V_0(A+B)^2}{\alpha^2 \hbar^2}}}{1 + 2n + \sqrt{(1+2l)^2 - \frac{2\mu V_0(A+B)^2}{\alpha^2 \hbar^2}}} \right)^2. \quad (15)$$

$$R(s) = NS^{\sqrt{\phi}}(1-S)^{\chi} P_n^{(2\sqrt{\phi}, \chi - \frac{1}{2})}(1-2S), \quad (16)$$

where

$$\chi = \frac{1}{2} + \frac{1}{2} \sqrt{(1+2l)^2 - \frac{2\mu V_0(A+B)^2}{\alpha^2 \hbar^2}}, \quad (17)$$

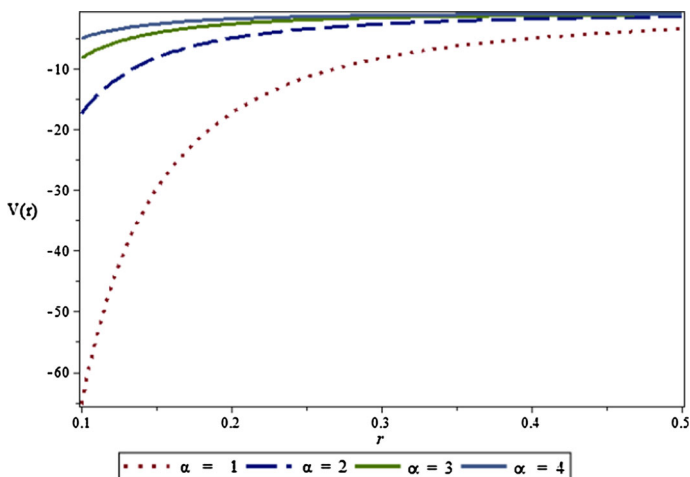
and N is the normalization constant.

4 Discussions

In Table 1, we have reported some numerical results of the energy levels for various values of $\alpha = 0.01, 0.02, 0.03$ and $\alpha = 0.04$, with $A = 0.4$, $B = 0.5$, $V_0 = 0.2$, $V_1 = 0.3$, $\mu =$

Table 1 Energy eigenvalues E_{nl} (in eV) of the modified Mobius square plus Kratzer potential with the parameters $V_0 = 0.2$, $V_1 = 0.3$, $A = 0.4$, $B = 0.5\mu = \hbar = 1$ for various values of $\alpha = 0.01, 0.02, 0.03$ and $\alpha = 0.04$

n	l	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$	$\alpha = 0.04$
0	0	-0.08003934750	-0.08015811930	-0.08035852857	-0.08064434923
1	0	-0.08003938225	-0.08015868825	-0.08036152045	-0.08065431289
	1	-0.08003943131	-0.08015950100	-0.08036587551	-0.08066919177
2	0	-0.08003940359	-0.08015907280	-0.08036384225	-0.08066339050
	1	-0.08003945333	-0.08015992935	-0.08036871494	-0.08068133306
	2	-0.08003950412	-0.08016085500	-0.08037441025	-0.08070419970
3	0	-0.08003941239	-0.08015932985	-0.08036616780	-0.08067558105
	1	-0.08003946309	-0.08016025000	-0.08037180454	-0.08069815483
	2	-0.08003951577	-0.08016130049	-0.08037900363	-0.08073017408
	3	-0.08003957043	-0.08016248327	-0.08038781830	-0.08077222805
4	0	-0.08003940999	-0.08015954750	-0.08036955474	-0.08069729128
	1	-0.08003946195	-0.08016055176	-0.08037622255	-0.08072629425
	2	-0.08003951712	-0.08016176864	-0.08038545604	-0.08077096722
	3	-0.08003957552	-0.08016320134	-0.08039734363	-0.08083230383
	4	-0.08003963715	-0.08016485308	-0.08041197681	-0.08091136228
5	0	-0.08003939819	-0.08015984573	-0.08037546617	-0.08073758103
	1	-0.08003945171	-0.08016095553	-0.08038345798	-0.08075510994
	2	-0.08003950998	-0.08016238159	-0.08039529184	-0.08083635188
	3	-0.08003957303	-0.08016412866	-0.08041110133	-0.08092283106
	4	-0.08003964088	-0.08016620157	-0.08043102470	-0.08103617189
	5	-0.08003971354	-0.08016860522	-0.08045520514	-0.08117810637

**Fig. 2** $V(r)$ against r with $V_0 = 2$, $A = -1$, $B = 2$, $V_1 = 0.1$

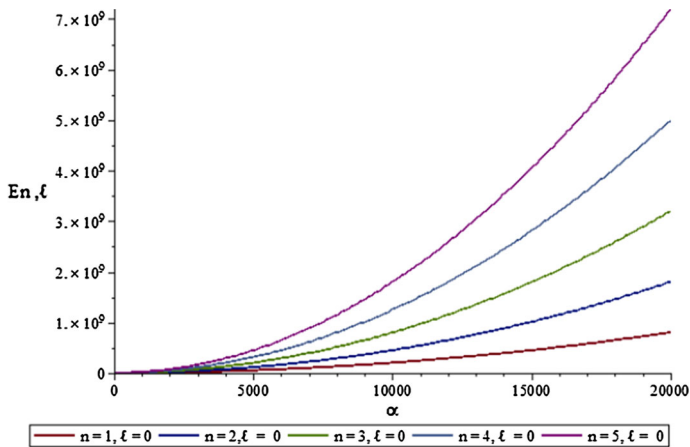


Fig. 3 E_{nl} against α for various n, ℓ , $V_0 = 3$, $A = -1$, $B = 2$, $V_1 = 0.1$, $\mu = \hbar = 1$, $\alpha = 0.1$

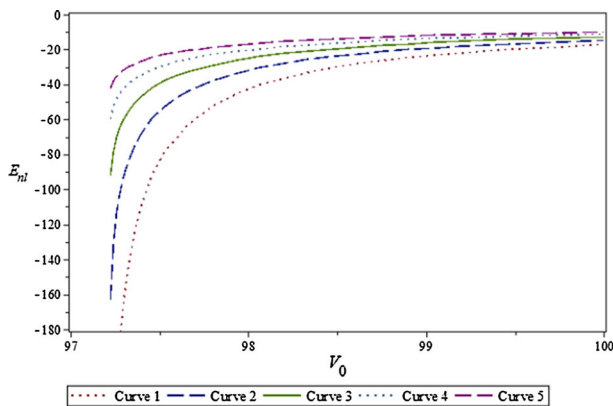


Fig. 4 E_{nl} against V_0 for $A = -1$, $B = 2$, $V_1 = 0.1$, $\mu = \hbar = 1$, $\alpha = 0.1$

1, $\hbar = 1$. In Fig. 1, a plot of the variation in the centrifugal term against the approximation is shown for two values of α . It is seen that the approximation is valid for $\alpha \ll 1$. Figure 2 shows the plot of the potential against the internuclear distance (r) for various values of the screening parameter. For higher values of r , the potential tends to be constant for each of the screening parameter. In Fig. 3, we examined the variation in the energy against the screening parameter. It is noted that as the screening parameter increases, the energy of the system increases monotonically from zero. It is equally seen that the energy of the system increases as the potential strength V_0 increases as shown in Fig. 4. A reverse case is shown in Fig. 5 when the energy is plotted against the potential strength V_1 . This indicates that when a particle is subjected to this system, the particle exhibits different features of V_0 and V_1 .

In Fig. 6, we showed the behavior of energy against A . An increase in A results in reduction in energy. Figure 7 shows the variation in energy against B . A reverse case of A is observed.

Special cases

Here, we examine special cases of Eq. (1) by adjusting some constants to obtain the following cases:

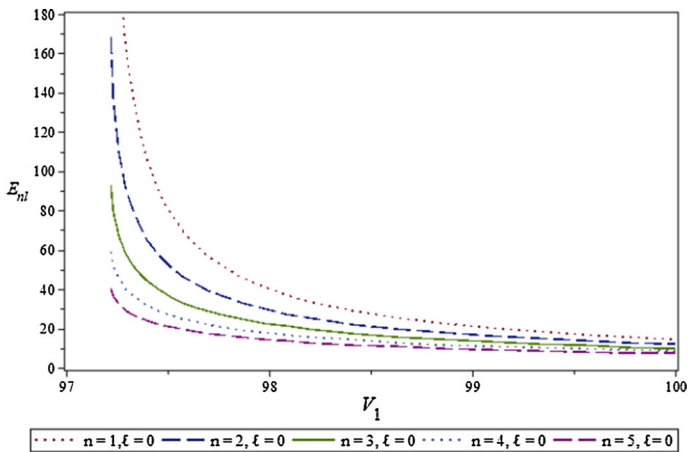


Fig. 5 E_{nl} against V_1 for $V_0 = 3$, $B = 2$, $V_1 = 0.1$, $\mu = \hbar = 1$, $\alpha = 0.1$

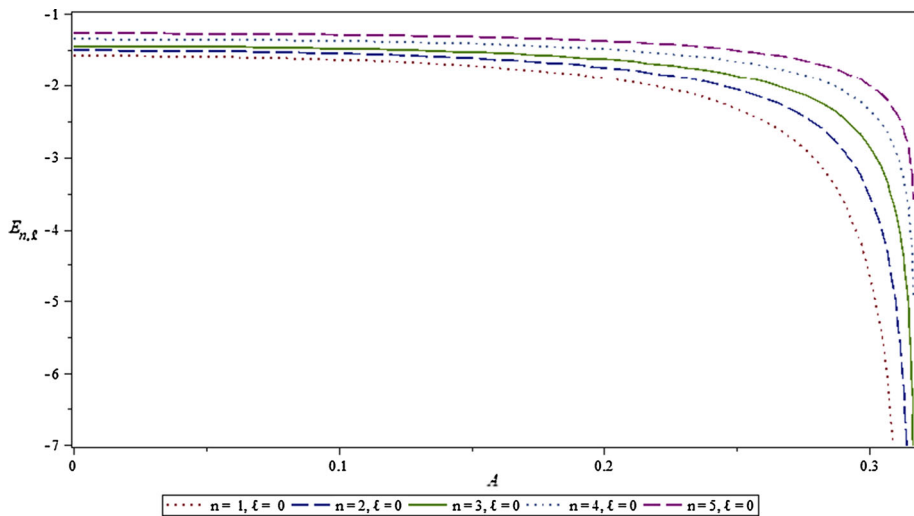


Fig. 6 E_{nl} against A for $V_0 = 3$, $B = 2$, $V_1 = 0.1$, $\mu = \hbar = 1$, $\alpha = 0.1$

Case I: Kratzer potential

By making $V_0 = 0$, Eq. (1) then reduces to the Kratzer potential of the form

$$V(r) = \frac{V_1 e^{-2\alpha r}}{1 - e^{-2\alpha r}}, \quad (18)$$

which turns the energy equation in Eq. (15) to become

$$E = -\frac{2\alpha^2 \hbar^2}{\mu} \left(\frac{\frac{\mu V_1}{2\alpha^2 \hbar^2} + (\ell + n + 1)^2}{2(\ell + n + 1)} \right)^2. \quad (19)$$

Case II: Modified Mobius square potential

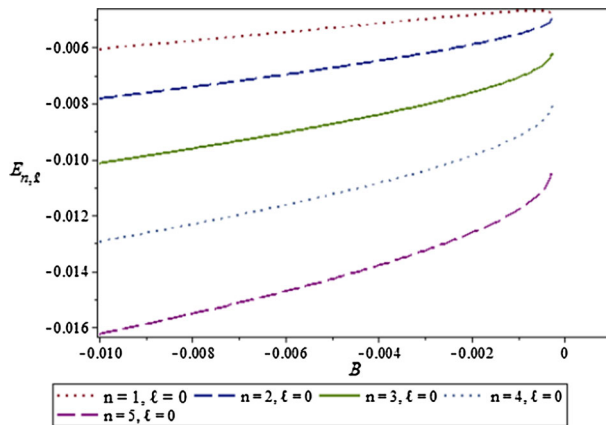


Fig. 7 E_{nl} against B for $A = -1$, $V_0 = 3$, $B = 2$, $V_1 = 0.1$, $\mu = \hbar = 1$, $\alpha = 0.1$

When the parameter $V_1 = 0$, Eq. (1) becomes the modified Mobius square potential as

$$V(r) = -V_0 \left(\frac{A + Be^{-2\alpha r}}{1 - e^{-2\alpha r}} \right)^2. \quad (20)$$

Hence, its energy eigenvalue equation can be obtained from Eq. (15) as

$$E = -AV_0 - \frac{2\alpha^2 \hbar^2}{\mu} \left(\frac{\frac{-\mu AV_0(2B+A)}{2\alpha^2 \hbar^2} + \frac{1}{2} + n(n+1) + \ell(\ell+1) + \left(n + \frac{1}{2}\right) \sqrt{(1+2\ell)^2 - \frac{2\mu V_0(A+B)^2}{\alpha^2 \hbar^2}}}{1 + 2n + \sqrt{(1+2\ell)^2 - \frac{2\mu V_0(A+B)^2}{\alpha^2 \hbar^2}}} \right)^2. \quad (21)$$

5 Conclusion

In this work, we have solved the Schrodinger equation with modified Mobius square plus Kratzer potential within the framework of the NU method. We obtained in detail the energy eigenvalues and the corresponding wave function. Special cases of the potential are also reported. Our results could find usefulness in both atomic and molecular physics.

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Author contributions CPO designed the work and drafted the manuscript; CAO performed the calculations and computation of results; OKE and AAI carried out the result analysis and discussions; HH proof read and approved the manuscript.

Compliance with ethical standards

Conflict of interest The authors declared that they have no conflict of interest.

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