



Research article

Out-of-plane equilibria in the perturbed photogravitational restricted three-body problem with Poynting-Robertson (P-R) drag



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ABSTRACT

We consider the primaries of the circular restricted three-body problem (CR3BP) to be luminous and study the effects of small perturbations in the Coriolis and centrifugal forces together with Poynting-Robertson (P-R) drag from both primaries on the motion of an infinitesimal body near the out-of-plane equilibrium points (OEPs). It is found that these points appear in pairs and, depending on the values of the parameters of the system, their number may be zero, two, $L_{6,7}$ or four, $L_{6,7,8,9}$. It is observed that the positions of these points depend on all the system parameters except small perturbation in the Coriolis force. This has been shown for binary systems RW-Monocerotis and Krüger-60. The linear stability of the out-of-plane equilibria is also studied and it is found that stability of some of these points significantly depends on the perturbing forces. Specifically, the motion of the infinitesimal body around the equilibria is conditionally stable only at points L_6 and L_7 in the absence of P-R drag effect in both binary systems. However, all the equilibria are unstable in the presence of the P-R drag effect. We may conclude therefore, that P-R effect destroys stability of the out-of-plane equilibria.

1. Introduction

The circular restricted three-body problem (CR3BP) consists of two finite bodies, known as primaries, which rotate in circular orbits around their common center of mass and a massless body which moves in the plane of motion of the primaries under their gravitational attraction and does not affect their motion. The CR3BP has been the well-known studied problem in Celestial Mechanics. It possesses five equilibrium points (EPs), three of which lie on the x -axis and are called collinear while the other two are away from the x -axis and are called triangular (non-collinear) equilibria. The three collinear points are generally unstable while the triangular points are stable for the mass ratio $\mu \leq 0.03850\dots$ (see, e.g., Szebehely, 1967). These EPs and the periodic orbits in the immediate neighborhood have enabled several space mission explorations (Sharer and Harrington, 1996; Farquhar, 2001; Giard and Montier, 2004) while some operations are still in progress.

During the past, several variants of this classical problem have been proposed by many scientists and astronomers in order to make it more realistic to real systems of dynamical astronomy. The dynamics of a small mass point around one star with a planet or two stars may be considered

as a generalization of the classical R3BP. In this case we have the so-called photogravitational restricted problem of three bodies. This special case of the CR3BP was firstly studied by Radzievskii (1950) and since then it has been extensively investigated (see, e.g., Simmons et al., 1985; Perdios et al., 2008; Papadakis, 1996; Perdios et al., 2015, among others). The significance of the effect of radiation on natural bodies or artificial satellites has been proved, recognized and used by many scientists, especially with respect to the solar sail of artificial satellites as well as to the formation of concentrations of interplanetary and interstellar dust in binary star systems (Gong et al., 2008).

The Doppler shift and absorptions as well as the subsequent re-emission of incident radiation, namely, the Poynting-Robertson's drag, are usually neglected in many research works for the approximation of radiation force, although the dissipative effects play a fundamental role in the dynamics of our Solar system. P-R effect is one of the most important mechanisms of dissipation which may be used in the investigation of the stability of zodiacal cloud, asteroidal particles as well as dust rings around planets. Many authors have discussed the effects of radiation pressure and P-R drag forces to the equilibrium points and their properties such as their linear stability (see, e.g., Ragos and Zafropoulos,

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1995; Lio et al., 1995; Kushvah, 2008; Rubican, 2013; Lhotka and Celletti, 2015; Vincent and Perdiou, 2021a, b; Vincent and Kalantonis, 2022).

In the photogravitational problem, in addition to the five coplanar points of the classical problem, there exist EPs which lie out of the orbital plane of the two primary bodies. The existence of such kind of EPs was pointed out by Radzievskii (1950, 1953) in the cases of Sun-Planet-particle and Galaxy Kernel-Sun-particle and he found two equilibrium points $L_{6,7}$ on the (x,z) plane symmetrically with respect to the (x,y) plane. The stability of these points was studied in the Solar problem by Perezhogin (1976, 1982) in the whole range of existence when the smaller body is considered non-luminous. Soon after, Simmons et al. (1985) proved the existence of a second family of the out-of-plane equilibrium points $L_{8,9}$, when both primary bodies are radiation sources and certain relations between the mass and the radiation pressure parameters hold. In the literature, several authors based their works on Radzievskii's model (see, e.g., Schuerman, 1980; Ragos and Zagouras, 1988; Roman, 2001; Vincent, 2019; Kalantonis et al., 2021) to understand various issues related to the dynamics of a particle around the OEPs when some radiation parameter is negative. The existence of these points is of particular astronomical interest in connection with planetary system formation, satellite motion, etc (Ragos et al., 1995).

Some significant contributions were made by Douskos and Markellos (2006), Shankaran et al. (2011), Singh (2013), Zeng et al. (2016), Nan et al. (2018), aiming to find the positions of out-of-plane equilibrium points or to examine the stability of motion around these points when one or both primaries are radiation source or have an oblate spheroid shape, as well as taking account of the influence of small perturbations in Coriolis and centrifugal forces in some cases.

Furthermore, when the P-R of one or both primaries is taken into account, the existence of OEPs, locating out of the orbital plane of primaries, is also declared (Chernikov, 1970; Ragos et al., 1995; Das et al., 2008, 2009; Singh and Aminu, 2015). Recently, Singh and Aminu (2014) investigated the existence and stability of triangular equilibrium points under the P-R effects of both luminous primaries together with small perturbations in the Coriolis and centrifugal forces. That investigation was performed in the two-dimensional case. The present work extends the research performed by Singh and Aminu (2014). We aim to examine effects of the system parameters on the existence, locations and stability of out of plane equilibrium points and to pursue some particular astronomical applications. To the present authors' knowledge, no work has been made of any equilibrium points out of the plane of motion of the primaries for this problem. As an application in this study, we consider the RW-Monocerotis and Krüger-60 binary systems. However, the labeling for the equilibrium points as L_i , $i = 6, 7, 8, 9$ is taken from Ragos et al. (1995).

The paper is organized as follows: In Section 2, we present the governing equations of motion for the system in the dissipative framework. In Section 3, we determine the existence and locations of the equilibrium points analytically and numerically, and verify them graphically for various values of the parameters under consideration, while their linear stability is analyzed in Section 4. Section 5 contains the numerical application, while Section 6 summarizes the discussion and conclusion of our study.

2. Equations of motion

We consider a barycentric coordinate system $Oxyz$ rotating relative to an inertial reference system with angular velocity ω about a common z -axis. Let the two massive bodies, with masses $m_1 = 1 - \mu$ and $m_2 = \mu$ ($0 < \mu = \frac{m_2}{(m_1+m_2)} \leq \frac{1}{2}$) where μ is the mass parameter, have fixed positions at the Ox -axis. We assume that the coordinate of the infinitesimal body is (x, y, z) and the two primaries m_1, m_2 are located at $(-\mu, 0)$ and $(1 - \mu, 0)$, respectively. Following Singh and Aminu (2014), the equations of

motion of the infinitesimal body under the P-R effects of both luminous primaries coupled with perturbations in the Coriolis and centrifugal forces, have the form:

$$\begin{aligned} \ddot{x} - 2a\dot{y} &= U_x - \frac{W_1}{r_1^2} \left[\frac{(x+\mu)}{r_1^2} \{ (x+\mu)\dot{x} + y\dot{y} + z\dot{z} \} + \dot{x} - y \right] \\ &\quad - \frac{W_2}{r_2^2} \left[\frac{(x+\mu-1)}{r_2^2} \{ (x+\mu-1)\dot{x} + y\dot{y} + z\dot{z} \} + \dot{x} - y \right], \\ \ddot{y} + 2a\dot{x} &= U_y - \frac{W_1}{r_1^2} \left[\frac{y}{r_1^2} \{ (x+\mu)\dot{x} + y\dot{y} + z\dot{z} \} + \dot{y} + (x+\mu) \right] \\ &\quad - \frac{W_2}{r_2^2} \left[\frac{y}{r_2^2} \{ (x+\mu-1)\dot{x} + y\dot{y} + z\dot{z} \} + \dot{y} + (x+\mu-1) \right], \\ \ddot{z} &= U_z - \frac{W_1}{r_1^2} \left[\frac{z}{r_1^2} \{ (x+\mu)\dot{x} + y\dot{y} + z\dot{z} \} + \dot{z} \right] \\ &\quad - \frac{W_2}{r_2^2} \left[\frac{z}{r_2^2} \{ (x+\mu-1)\dot{x} + y\dot{y} + z\dot{z} \} + \dot{z} \right] \end{aligned} \quad (1)$$

where

$$U_x = \beta x - \frac{Q_1(x+\mu)}{r_1^3} - \frac{Q_2(x+\mu-1)}{r_2^3},$$

$$U_y = \beta y - \frac{Q_1 y}{r_1^3} - \frac{Q_2 y}{r_2^3},$$

$$U_z = -\frac{Q_1 z}{r_1^3} - \frac{Q_2 z}{r_2^3},$$

with

$$U = \frac{\beta(x^2 + y^2)}{2} + \frac{Q_1}{r_1} + \frac{Q_2}{r_2},$$

and

$$r_1^2 = (x+\mu)^2 + y^2 + z^2, \quad r_2^2 = (x+\mu-1)^2 + y^2 + z^2, \quad Q_1 = q_1(1-\mu), \\ Q_2 = q_2\mu,$$

$$W_1 = \frac{(1-\mu)(1-q_1)}{c_d}, \quad W_2 = \frac{\mu(1-q_2)}{c_d} \quad (2)$$

Here the parameters q_1, q_2 ($q_i \leq 1, i = 1, 2$) involved in Eq. (2) are radiation pressure of the bigger m_1 and smaller m_2 primary, respectively, $r_i, i = 1, 2$ are the distances of the third body from the two primaries m_1 and m_2 , respectively, while α and β represent parameters through which small perturbations are given in the Coriolis and centrifugal forces, respectively, such that $\alpha = 1 + \varepsilon$, $\beta = 1 + \varepsilon'$, $\varepsilon\varepsilon' \ll 1$. Also, W_1, W_2 ($W_i < 1, i = 1, 2$) stand for P-R drag of the first and second primary, respectively, c_d is the non-dimensional velocity of light while the dots denote differentiation with respect to time t . Notice that when radiation force is absent, there will be no P-R drag force.

3. Existence and location of out-of-plane equilibrium points

The equilibrium (or Lagrangian) points are those points at which the velocity and acceleration of the infinitesimal body are zero. A study of the triangular equilibria, and their stability on the plane Oxy of the problem has been performed in Singh and Aminu (2014). This present paper investigates the equilibria out of this plane. Thus, the equilibria can be determined from the equations of motion (1) by setting

$$\dot{x} = \dot{y} = \dot{z} = \ddot{x} = \ddot{y} = \ddot{z} = 0, \quad (3)$$

and solving the resulting system for $x, y, z \neq 0$. Applying these conditions (3) in Eq. (1), we obtain:

$$\beta x - \frac{Q_1(x+\mu)}{r_1^3} - \frac{Q_2(x+\mu-1)}{r_2^3} + \left[\frac{W_1}{r_1^2} + \frac{W_2}{r_2^2} \right] y = 0, \quad (4)$$

$$\left[\beta - \frac{Q_1}{r_1^3} - \frac{Q_2}{r_2^3} \right] y - \frac{W_1}{r_1^2} (x+\mu) - \frac{W_2}{r_2^2} (x+\mu-1) = 0, \quad (5)$$

$$-\left[\frac{Q_1}{r_1^3} + \frac{Q_2}{r_2^3} \right] z = 0. \quad (6)$$

For $z \neq 0$, it is easy to see from Eq. (6) that

$$\frac{r_2}{r_1} = \left(-\frac{Q_2}{Q_1} \right)^{\frac{1}{3}} \quad (7)$$

where by, Eq. (5) leads to the condition $y \neq 0$ due to the presence of the P-R drag. Obviously, from Eq. (7) it can be seen that, for the existence of any real solution the following condition as expressed in Eq. (8):

$$q_1, q_2 < 0 \quad (8)$$

is necessary to hold; which means that the radiation pressure force of just one of the bodies dominates the gravitational one.

Therefore, we study the equilibria that lay outside of the plane of the primaries in two cases, as described below. First, when the Poynting-Robertson (P-R) effect is neglected and second when the P-R drag effect of the radiating primaries is incorporated.

3.1. Location and stability of out of plane equilibria in the absence of P-R drag effect

In the absence of P-R drag (i.e. $W_1 = W_2 = 0$), we observe that the second Eq. (5) is satisfied for $y = 0$ and we solve the remaining two equations for $x \neq 0$ and $z \neq 0$. So, we obtain the coordinates (x_0, z_0) of equilibrium points as solutions of the equations:

$$\beta x - \frac{Q_1(x+\mu)}{r_1^3} - \frac{Q_2(x+\mu-1)}{r_2^3} = 0, \quad (9)$$

$$\left[\frac{Q_1}{r_1^3} + \frac{Q_2}{r_2^3} \right] z = 0 \quad (10)$$

From Eq. (10) and the definition of r_1, r_2 , we have that

$$x_0 = \frac{1}{2} - \mu + \frac{1}{2} \left[1 - \left(-\frac{Q_2}{Q_1} \right)^{\frac{2}{3}} \right] r_{01}^2 \quad (11)$$

Using Eqs. (10) and (11) in Eq. (9) we have that the distance of any existing equilibrium point from the bigger primary body m_1 (see Figure 1a, b)) must satisfy the equation

$$p(r_{01}) = ar_{01}^5 + br_{01}^3 + c = 0, \quad (12)$$

with

$$a = \frac{\beta}{2} \left[1 - \left(-\frac{Q_2}{Q_1} \right)^{\frac{2}{3}} \right], \quad b = \beta \left(\frac{1}{2} - \mu \right), \quad c = -Q_1,$$

where $r_{01}^2 = (x_0 + \mu)^2 + z_0^2$ and $r_{02}^2 = (x_0 + \mu - 1)^2 + z_0^2$ are the distances between the infinitesimal body and bigger and smaller stars, respectively.

From Eq. (12), any positive root together with the corresponding value of r_{02} obtained from Eq. (10) must fulfill the triangular inequality as expressed in Eq. (13):

$$r_{01} + r_{02} > 1 \quad (13)$$

Using $r_{01}^2 = (x_0 + \mu)^2 + z_0^2$, the z_0 -coordinates can be expressed as

$$z_0 = \pm \sqrt{r_{01}^2 - (x_0 + \mu)^2} \quad (14)$$

Eqs. (11) and (14) are the coordinates of the out of plane equilibria of the model-problem without P-R drag effect. It is interesting to note that the coordinates are in agreement with the numerically determined exact positions of the out of plane equilibria computed in Kalantonis et al. (2021) upon substituting $\beta = \omega^2$ in our potential-like function. We found that for the range of values of β, q_1, q_2 , and μ the problem may admit up to four equilibrium points in the xz -plane in symmetrical positions with respect to the xy -plane. In particular, critical points L_6, L_8 which lie in the positive half-plane and their symmetric points with respect to the Oxy plane, L_7, L_9 on the negative half-plane. In Figure 1a, we illustrate the position of the two critical points, namely $L_{6(7)}$ for $\mu = 0.287, q_1 = -0.04, q_2 = 0.3$ and $\epsilon' = 0.02$, which we have found by solving numerically the Eqs. (9) and (10). In the same figure (Figure 1b), we present the location of the four critical points, namely $L_{6(7)}$ and $L_{8(9)}$, for $\mu = 0.287, q_1 = 0.5, q_2 = -1.35$ and $\epsilon' = 0.01$. One can easily see equilibria on the xz -plane given by the mutual intersections of the curves, Eq. (9) (orange line in the figure) and Eq. (10) (light blue line in the figure). The two black points are the positions of the primary bodies and the green dots are the positions of the equilibria.

From the plot (see Figure 1a, b), we observe that for certain values of the parameters involved in this problem, there is one or two pairs of equilibrium points in symmetrical positions with respect to the orbital plane. Similar phenomenon we observe in the classical photogravitational three-body problem where for certain values of the radiation factors, there is one or two pairs of equilibria, symmetrical with respect to the orbital plane and lying on a plane perpendicular to the orbital plane and containing the primaries (see e.g., Simmons et al., 1985). Note that equilibria L_8, L_9 are farther from the origin whereas the other two points L_6, L_7 on the plane are closer to the Ox -axis.

3.2. Location and stability of out of plane equilibria in the presence of P-R drag effect

In this premise, the out of plane equilibria of the primaries lay on the $Oxyz$ plane, and with the help of Eq. (4) and the definition of r_1, r_2 , we obtain the positions of the equilibrium points (x_0, y_0, z_0) as follows (Ragos et al., 1995):

$$x_0 = \frac{1}{2} - \mu + \frac{1}{2} \left[1 - \left(-\frac{Q_2}{Q_1} \right)^{\frac{2}{3}} \right] r_{01}^2 \quad (15)$$

Using the value of x_0 from (15) in (5) together with help of (7), we obtain

$$y_0 = \frac{1}{2\beta} \left\{ W_1 - W_2 - \left[W_1 \left(-\frac{Q_2}{Q_1} \right)^{\frac{2}{3}} - W_2 \left(-\frac{Q_1}{Q_2} \right)^{\frac{2}{3}} \right] \right\} + \frac{1}{2\beta} \left[W_1 - W_2 \left(-\frac{Q_1}{Q_2} \right)^{\frac{2}{3}} \right] \frac{1}{r_{01}^2} \quad (16)$$

Therefore, by combining Eqs. (6), (15), and (16) with Eq. (4), we obtain for the out of plane equilibrium points:

$$p(r_{01}) = a_6 r_{01}^6 + a_4 r_{01}^4 + a_2 r_{01}^2 + a_1 r_{01} + a_0 = 0, \quad (17)$$

bearing in mind that the distance from any existing equilibrium point to the bigger primary m_1 must satisfy the sixth-degree polynomial equation in r_{01} where we have abbreviated the quantities:

$$a_6 = \frac{\beta}{2} \left[1 - \left(-\frac{Q_2}{Q_1} \right)^{\frac{2}{3}} \right], \quad a_4 = \beta \left(\frac{1}{2} - \mu \right),$$

$$a_2 = \frac{1}{2\beta} \left[W_1 + W_2 \left(-\frac{Q_1}{Q_2} \right)^{\frac{2}{3}} \right] \left\{ W_1 - W_2 - \left[W_1 \left(-\frac{Q_2}{Q_1} \right)^{\frac{2}{3}} - W_2 \left(-\frac{Q_1}{Q_2} \right)^{\frac{2}{3}} \right] \right\},$$

$$a_1 = -Q_1,$$

$$a_0 = \frac{1}{2\beta} \left[W_1^2 - \left(-\frac{Q_1}{Q_2} \right)^{\frac{4}{3}} W_2^2 \right]$$

where $r_{01}^2 = (x_0 + \mu)^2 + y_0^2 + z_0^2$ and $r_{02}^2 = (x_0 + \mu - 1)^2 + y_0^2 + z_0^2$ are the distances between the infinitesimal body and bigger and smaller stars, respectively.

From Eq. (17), any positive root together with the corresponding value of r_{02} obtained from Eq. (6) must fulfill the triangle condition as expressed in Eq. (18):

$$r_{01} + r_{02} > 1 \quad (18)$$

Using $r_{01}^2 = (x_0 + \mu)^2 + y_0^2 + z_0^2$, the z_0 -coordinates can be expressed as

$$z_0 = \pm \sqrt{r_{01}^2 - (x_0 + \mu)^2 - y_0^2} \quad (19)$$

Eqs. (15), (16), and (19) yield coordinates of the OEPs of the problem under investigation. In the case $W_1 = 0 = W_2$ they coincide with Eqs. (11) and (14). It is seen that the equilibrium positions are affected by the radiation pressure, P-R drag and small perturbation in the centrifugal force, but unaffected by that of Coriolis force. Evidently, when $\beta = 1$, the coordinates are fully analogous with those obtained by Ragos et al. (1995). Furthermore, in Figure 2 we give the position diagram of Ragos et al. (1995) for two pairs of out of plane equilibrium points when $\pm = 0.25$ and $\beta = 1$ with variation in radiation factors, which was reproduced here for checking purposes and for comparison. Figure 2a depicts the variation of $\times 10^{-5}$ versus x -coordinate, Figure 2b presents the variation of z versus x -coordinate while Figure 2c shows the variation of x -coordinate versus q_2 . For this special case, the numerical solution depicted in Figure 2 agrees with the result presented in Ragos et al. (1995) (see Figure 3(a, b) and Figure 4 of that paper). Note the different scales for the horizontal and the vertical axes are due to the machine precision.

4. Linear stability of the out-of-plane equilibrium points

In order to study the stability of the out of plane equilibrium points $L_{6,7}$ and $L_{8,9}$ where the two points $L_{6,7}$ ($L_{8,9}$) are symmetrical with respect to the x -axis, we transfer the origin to $L_6(x_0, y_0, \pm z_0)$ and $L_8(x_0, y_0, \pm z_0)$ by introducing the new variables (ξ, η, ζ) and linearize the equations of motion (1) to first order terms arriving at the variational Eq. (20) in compact form (see also Ragos et al., 1995; Singh and Amuda, 2015) as follows:

$$\ddot{\xi} - 2a\dot{\eta} = A_2\xi + A_4\eta + A_6\zeta + A_1\dot{\xi} + A_3\dot{\eta} + A_5\dot{\zeta},$$

$$\ddot{\eta} + 2a\dot{\xi} = B_2\xi + B_4\eta + B_6\zeta + B_1\dot{\xi} + B_3\dot{\eta} + B_5\dot{\zeta}, \quad (20)$$

$$\ddot{\zeta} = C_2\xi + C_4\eta + C_6\zeta + C_1\dot{\xi} + C_3\dot{\eta} + C_5\dot{\zeta}$$

where we have set for abbreviation the quantities A_i, B_i, C_i :

$$A_1 = \Omega_{xx}^{(0)} = - \left[\frac{W_1(x_0 + \mu)^2}{r_{01}^4} + \frac{W_2(x_0 + \mu - 1)^2}{r_{02}^4} + \frac{W_1}{r_{01}^2} + \frac{W_2}{r_{02}^2} \right],$$

$$A_2 = \Omega_{xx}^{(0)} = \beta - \frac{Q_1}{r_{01}^3} - \frac{Q_2}{r_{02}^3} + 3 \left[\frac{Q_1(x_0 + \mu)^2}{r_{01}^5} + \frac{Q_2(x_0 + \mu - 1)^2}{r_{02}^5} \right]$$

$$- 2 \left[\frac{W_1(x_0 + \mu)}{r_{01}^4} + \frac{W_2(x_0 + \mu - 1)}{r_{02}^4} \right] y_0,$$

$$A_3 = \Omega_{xy}^{(0)} = - \left[\frac{W_1(x_0 + \mu)}{r_{01}^4} + \frac{W_2(x_0 + \mu - 1)}{r_{02}^4} \right] y_0,$$

$$A_4 = \Omega_{xy}^{(0)} = 3 \left[\frac{Q_1(x_0 + \mu)}{r_{01}^5} + \frac{Q_2(x_0 + \mu - 1)}{r_{02}^5} \right] y_0 + \frac{W_1}{r_{01}^2} + \frac{W_2}{r_{02}^2}$$

$$- 2 \left[\frac{W_1}{r_{01}^4} + \frac{W_2}{r_{02}^4} \right] y_0^2,$$

$$A_5 = \Omega_{xz}^{(0)} = - \left[\frac{W_1(x_0 + \mu)}{r_{01}^4} + \frac{W_2(x_0 + \mu - 1)}{r_{02}^4} \right] z_0,$$

$$A_6 = \Omega_{xz}^{(0)} = 3 \left[\frac{Q_1(x_0 + \mu)}{r_{01}^5} + \frac{Q_2(x_0 + \mu - 1)}{r_{02}^5} \right] z_0 - 2 \left[\frac{W_1}{r_{01}^4} + \frac{W_2}{r_{02}^4} \right] y_0 z_0,$$

$$B_1 = \Omega_{yx}^{(0)} = - \left[\frac{W_1(x_0 + \mu)}{r_{01}^4} + \frac{W_2(x_0 + \mu - 1)}{r_{02}^4} \right] y_0,$$

$$B_2 = \Omega_{yx}^{(0)} = 3 \left[\frac{Q_1(x_0 + \mu)}{r_{01}^5} + \frac{Q_2(x_0 + \mu - 1)}{r_{02}^5} \right] y_0 - \frac{W_1}{r_{01}^2} - \frac{W_2}{r_{02}^2}$$

$$+ 2 \left[\frac{W_1(x_0 + \mu)^2}{r_{01}^4} + \frac{W_2(x_0 + \mu - 1)^2}{r_{02}^4} \right],$$

$$B_3 = \Omega_{yy}^{(0)} = - \frac{W_1}{r_{01}^2} - \frac{W_2}{r_{02}^2} - \left[\frac{W_1}{r_{01}^4} + \frac{W_2}{r_{02}^4} \right] y_0^2, \quad (21)$$

$$B_4 = \Omega_{yy}^{(0)} = \beta - \frac{Q_1}{r_{01}^3} - \frac{Q_2}{r_{02}^3} + 3 \left[\frac{Q_1}{r_{01}^5} + \frac{Q_2}{r_{02}^5} \right] y_0^2$$

$$+ 2 \left[\frac{W_1(x_0 + \mu)}{r_{01}^4} + \frac{W_2(x_0 + \mu - 1)}{r_{02}^4} \right] y_0,$$

$$B_5 = \Omega_{yz}^{(0)} = - \left[\frac{W_1}{r_{01}^4} + \frac{W_2}{r_{02}^4} \right] y_0 z_0,$$

$$B_6 = \Omega_{yz}^{(0)} = 3 \left[\frac{Q_1}{r_{01}^5} + \frac{Q_2}{r_{02}^5} \right] y_0 z_0 + 2 \left[\frac{W_1(x_0 + \mu)}{r_{01}^4} + \frac{W_2(x_0 + \mu - 1)}{r_{02}^4} \right] z_0,$$

$$C_1 = \Omega_{zx}^{(0)} = - \left[\frac{W_1(x_0 + \mu)}{r_{01}^4} + \frac{W_2(x_0 + \mu - 1)}{r_{02}^4} \right] z_0,$$

$$C_2 = \Omega_{zx}^{(0)} = 3 \left[\frac{Q_1(x_0 + \mu)}{r_{01}^5} + \frac{Q_2(x_0 + \mu - 1)}{r_{02}^5} \right] z_0,$$

$$C_3 = \Omega_{zy}^{(0)} = - \left[\frac{W_1}{r_{01}^4} + \frac{W_2}{r_{02}^4} \right] y_0 z_0,$$

$$C_4 = \Omega_{zy}^{(0)} = 3 \left[\frac{Q_1}{r_{01}^5} + \frac{Q_2}{r_{02}^5} \right] y_0 z_0,$$

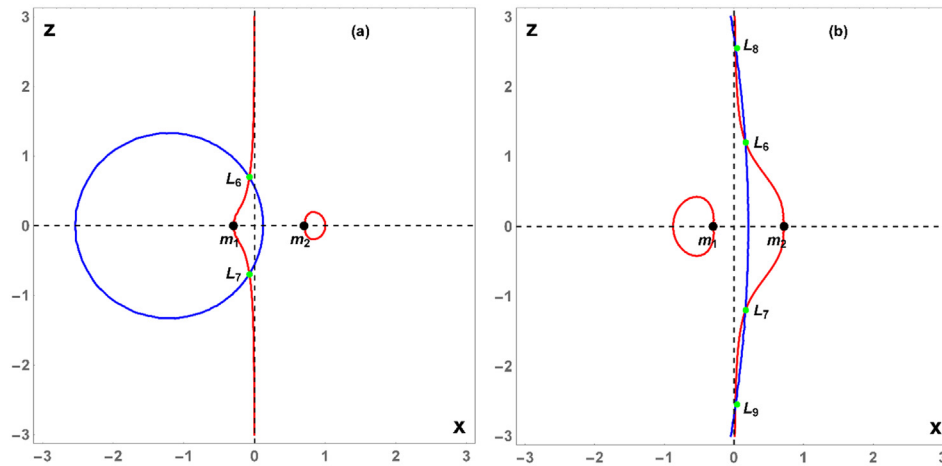


Figure 1. Panel (a) Positions of the two equilibria $L_{6(7)}$ for $q_1 = -0.04, q_2 = 0.3$ and $\epsilon' = 0.02$, Panel (b) Positions of the four critical $L_{6(7)}$ and $L_{8(9)}$ ones, for $q_1 = 0.5, q_2 = -1.35$ and $\epsilon' = 0.01$. Note that green and black dots denote positions of the equilibria and primaries, respectively. The mass parameter, for both cases, is $\mu = 0.287$.

$$C_5 = \Omega_{zz}^{(0)} = -\frac{W_1}{r_{01}^2} - \frac{W_2}{r_{02}^2} - \left[\frac{W_1}{r_{01}^4} + \frac{W_2}{r_{02}^4} \right] z_0^2,$$

$$C_6 = \Omega_{zz}^{(0)} = -\frac{Q_1}{r_{01}^3} - \frac{Q_2}{r_{02}^3} + 3 \left[\frac{Q_1}{r_{01}^5} + \frac{Q_2}{r_{02}^5} \right] z_0^2,$$

with

$$r_{01}^2 = (x_0 + \mu)^2 + y_0^2 + z_0^2, \quad r_{02}^2 = (x_0 + \mu - 1)^2 + y_0^2 + z_0^2 \quad (22)$$

where we have denoted the right-hand side of Eq. (1) by $\Omega_x = \frac{\partial \Omega}{\partial x}$, $\Omega_y = \frac{\partial \Omega}{\partial y}$ and $\Omega_z = \frac{\partial \Omega}{\partial z}$, respectively, $\pm A_i, B_i, C_i, i = 1, 2, \dots, 6$ are constants with dots being the derivatives with respect to time t . In Eq. (21) the second order partial derivatives of Ω are denoted by subscripts, while the superscript

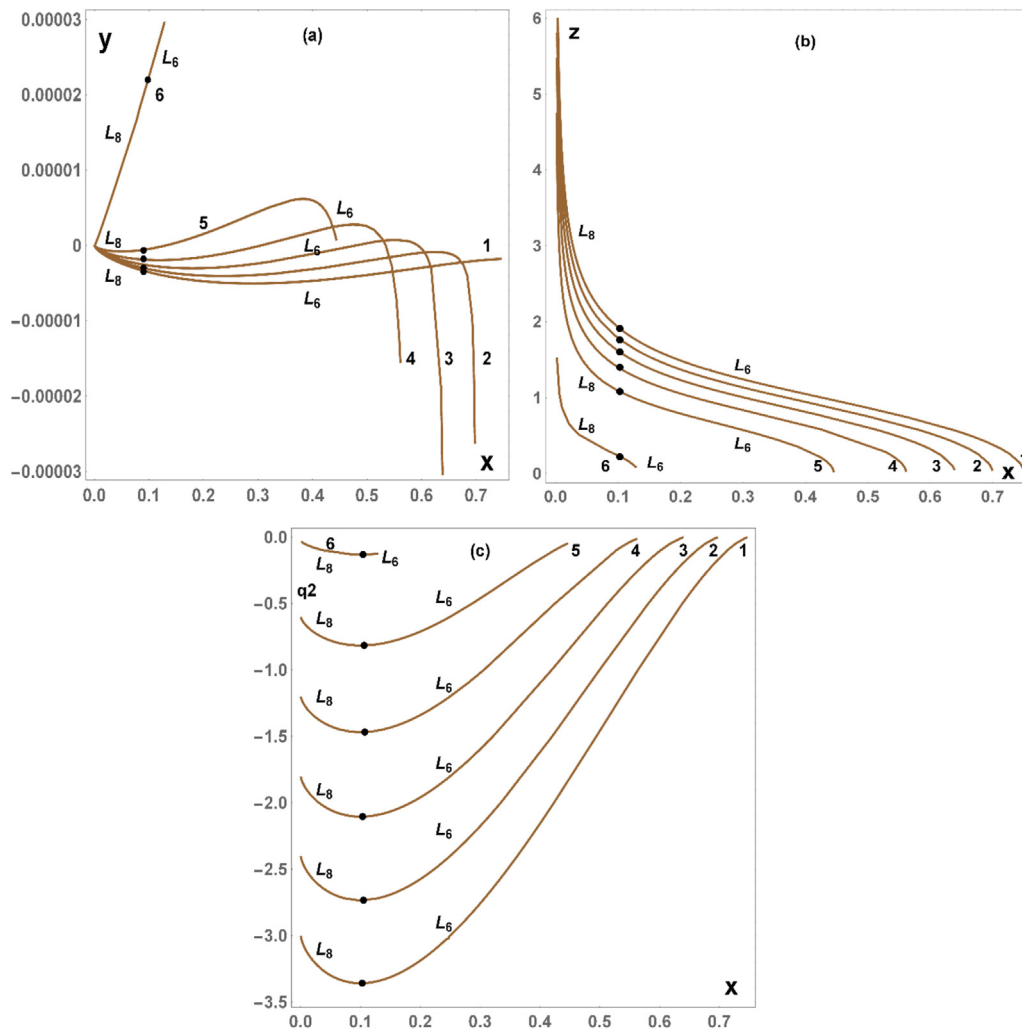


Figure 2. Positions of the two positive out of plane equilibrium points L_6, L_8 for binary system Kruger 60 ($\mu = 0.25$) as a function radiation factor q_2 in the interval $-3.5 \leq q_2 < 0$ for various values of the radiation factor q_1 : (1) $q_1 = 1$, (2) $q_1 = 0.8$, (3) $q_1 = 0.6$, (4) $q_1 = 0.4$, (5) $q_1 = 0.2$, (6) $q_1 = 0.01$. (a) Variation of y versus x -coordinate and (b) variation of z versus x -coordinate and (c) variation of x -coordinate versus q_2 . Note: the values of $\mu = 0.25$ and $C_d = 48002.33$ are fixed in all cases (Ragos et al., 1995). The different scales for the horizontal and the vertical axes are due to the machine precision.

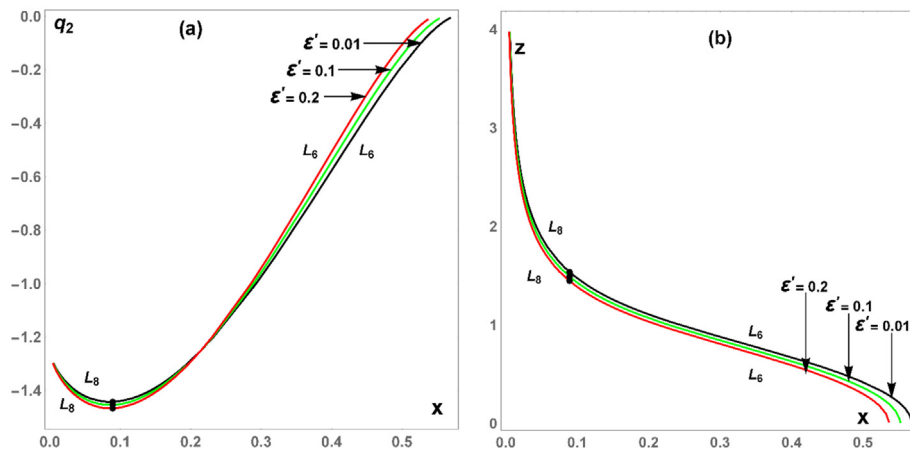


Figure 3. Variation of the equilibrium solutions for L_6 and L_8 (the situation is same at the symmetric point L_7 and L_9 , respectively) of RW-Monocerotis ($\mu = 0.2857$) system for (a) x – coordinate vs. q_2 , and (b) x – vs. z – coordinate as a function of the parameter q_2 in the interval $[-1.5, 0]$ for $q_1 = 0.5$ and for different values of the centrifugal parameter $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line) and $\epsilon' = 0.2$ (red line), respectively. Note that the symbol $\bullet$ represents where L_6 coincides with L_8 .

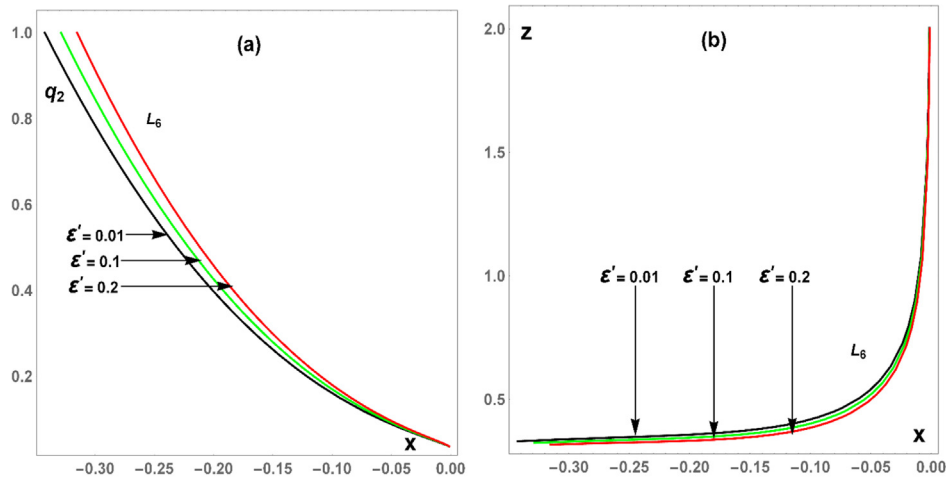


Figure 4. Variation of the equilibrium solutions for L_6 (the situation is same at the symmetric point L_7) of Krüger-60 ($\mu = 0.3721$) system for (a) x – coordinate vs. q_2 , and (b) x – vs. z – coordinate as a function of the parameter q_2 in the interval $(0, 1]$ for $q_1 = -0.02$ and for different values of the centrifugal parameter $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line) and $\epsilon' = 0.2$ (red line), respectively.

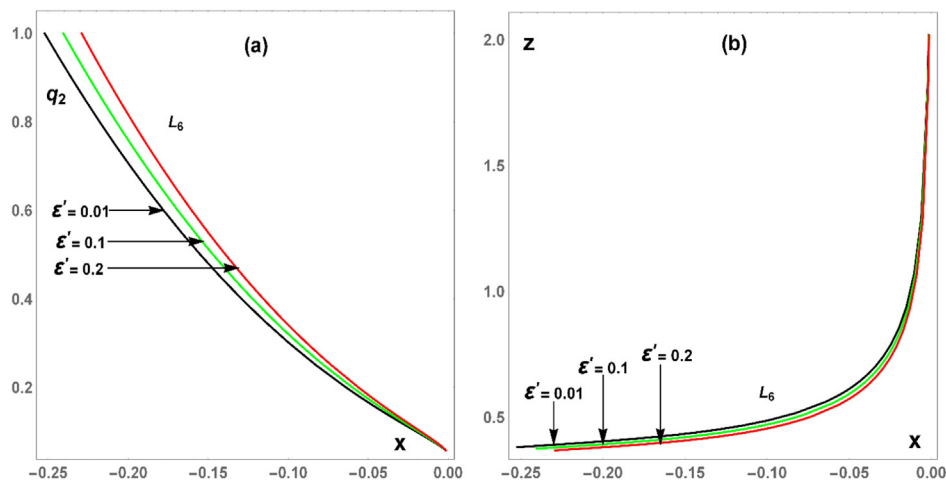


Figure 5. Variation of the equilibrium solutions for L_6 (the situation is same at the symmetric point L_7) of RW-Monocerotis ($\mu = 0.2857$) system for (a) x – coordinate vs. q_2 , and (b) x – vs. z – coordinate as a function of the parameter q_2 in the interval $(0, 1]$ for $q_1 = -0.02$ and for different values of the centrifugal parameter $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line) and $\epsilon' = 0.2$ (red line), respectively.

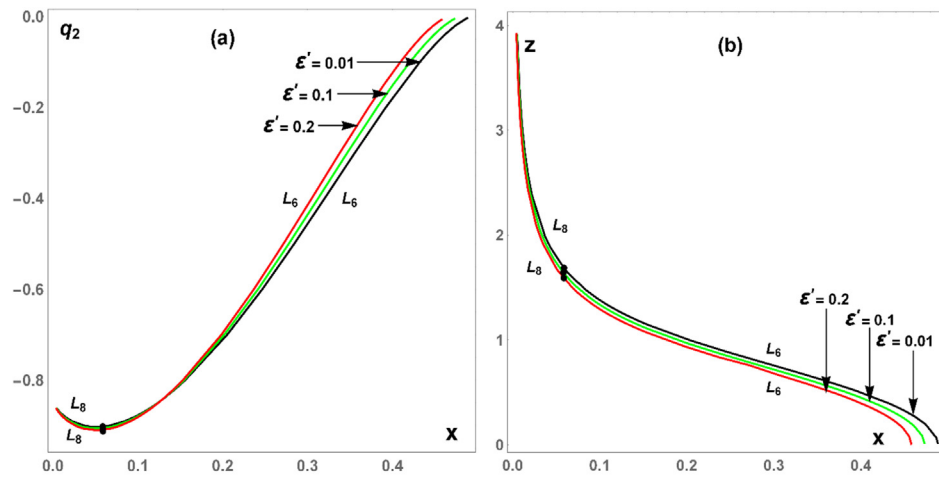


Figure 6. Variation of the equilibrium solutions for L_6 and L_8 (the situation is same at the symmetric point L_7 and L_9 , respectively) of Krüger-60 ($\mu = 0.3721$) system for (a) $x-$ coordinate vs. q_2 , and (b) $x-$ vs. $z-$ coordinate as a function of the parameter q_2 in the interval $[-0.91, 0]$ for $q_1 = 0.5$ and for different values of the centrifugal parameter $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line) and $\epsilon' = 0.2$ (red line), respectively. Note that the symbol $\cdot$ represents where L_6 coincides with L_8 .

(0) denotes evaluation at the coordinates $(x_0, y_0, \pm z_0)$ of the out of plane equilibrium points. We also note that r_{01} and r_{02} in Eq. (22) are the distances given by Eq. (2) at the equilibrium.

The characteristic equation corresponding to system (20) is given as:

$$\lambda^6 + f_5\lambda^5 + f_4\lambda^4 + f_3\lambda^3 + f_2\lambda^2 + f_1\lambda + f_0 = 0, \quad (23)$$

where

$$f_5 = -(A_1 + B_3 + C_5),$$

$$f_4 = -(A_2 + A_3B_1 - A_1B_3 + B_4 + A_5C_1 + B_5C_3 - A_1C_5 - B_3C_5 + C_6 - 2\alpha A_3 + 2B_1\beta - 4\alpha^2),$$

$$f_3 = -(A_4B_1 + A_3B_2 - A_2B_3 - A_1B_4 + A_6C_1 - A_5B_3C_1 + A_3B_5C_1 + A_5C_2 + A_5B_1C_3 - A_1B_5C_3 + B_6C_3 + B_5C_4 - A_2C_5 - A_3B_1C_5 + A_1B_3C_5 - B_4C_5 - A_1C_6 - B_3C_6 - 2\alpha A_4 + 2\alpha B_2 + 2\alpha B_5C_1 - 2\alpha A_5C_3 + 2\alpha A_3C_5 - 2\alpha B_1C_5 + 4\alpha^2C_5), \quad (24)$$

$$f_2 = (-A_4B_2 + A_2B_4 + A_6B_3C_1 + A_5B_4C_1 - A_4B_5C_1 - A_3B_6C_1 - A_6C_2 + A_5B_3C_2 - A_3B_5C_2 - A_6B_1C_3 - A_5B_2C_3 + A_2B_5C_3 + A_1B_6C_3 - A_5B_1C_4 + A_1B_5C_4 - B_6C_4 + A_4B_1C_5 + A_3B_2C_5 - A_2B_3C_5 - A_1B_4C_5 + A_2C_6 + A_3B_1C_6 - A_1B_3C_6 + B_4C_6 - 2\alpha B_6C_1 - 2\alpha B_5C_2 + 2\alpha A_6C_3 + 2\alpha A_5C_4 - 2\alpha A_4C_5 + 2\alpha B_2C_5 - 2\alpha A_3C_6 + 2\alpha B_1C_6 - 4\alpha^2C_6),$$

$$f_1 = (A_6B_4C_1 - A_4B_6C_1 + A_6B_3C_2 + A_5B_4C_2 - A_4B_5C_2 - A_3B_6C_2 - A_6B_2C_3 + A_2B_6C_3 - A_6B_1C_4 - A_5B_2C_4 + A_2B_5C_4 + A_1B_6C_4 + A_4B_2C_5 - A_2B_4C_5 + A_4B_1C_6 + A_3B_2C_6 - A_2B_3C_6 - A_1B_4C_6 - +2\alpha B_6C_2 + 2\alpha A_6C_4 - 2\alpha A_4C_6 + 2\alpha B_2C_6),$$

$$f_0 = A_6B_4C_2 - A_4B_6C_2 - A_6B_2C_4 + A_2B_6C_4 + A_4B_2C_6 - A_2B_4C_6$$

which is a polynomial of sixth degree in λ and $f_i, i = 0, 1, \dots, 5$ are arbitrary constants. The eigenvalues of the characteristic Eq. (23) determine the stability or instability of the respective equilibrium points. According to the Lyapunov theorem, stability of the equilibrium points is established if we obtain only negative real numbers or only imaginary roots or if the real part of complex roots is negative. In other case the equilibrium will be unstable.

Table 1. The eigenvalues $\lambda_{1,2}, \lambda_{3,4}, \lambda_{5,6}$ of Eq. (26) and the corresponding positions $(x_0, \pm z_0)$ of the out-of-plane equilibria $L_{6,7}$ and $L_{8,9}$ for RW-Monocerotis in the absence of P-R.

q_1	q_2	ϵ'	ϵ	$L_{6,7}$	$L_{8,9}$
0.49	- 1.36	0.01	0.02	$\pm 0.36026i, \pm 0.58480i, \pm 1.29222i$ (x_0, z_0) = (0.151759, ± 1.24205)	$\pm 0.067672i, \pm 0.83445i, \pm 1.20411i$ (x_0, z_0) = (0.0299995, ± 2.2384)
0.489	- 1.359	0.03	0.04	$\pm 0.37887i, \pm 0.55316i, \pm 1.34791i$ (x_0, z_0) = (0.151893, ± 1.23156)	$\pm 0.068683i, \pm 0.80368i, \pm 1.27484i$ (x_0, z_0) = (0.0299063, ± 2.22433)
0.488	- 1.358	0.05	0.06	$\pm 0.40440i, \pm 0.51684i, \pm 1.40134i$ (x_0, z_0) = (0.152016, ± 1.22133)	$\pm 0.069697i, \pm 0.78061i, \pm 1.33787i$ (x_0, z_0) = (0.0298226, ± 2.21042)
0.487	- 1.357	0.07	0.075	$\pm 0.43703i, \pm 0.48405i, \pm 1.4343i$ (x_0, z_0) = (0.152127, ± 1.21138)	$\pm 0.070711i, \pm 0.77459i, \pm 1.37387i$ (x_0, z_0) = (0.029746, ± 2.19669)

Table 2. The eigenvalues $\lambda_{1,2}, \lambda_{3,4}, \lambda_{5,6}$ of Eq. (26) and the corresponding positions $(x_0, \pm z_0)$ of the out-of-plane equilibria $L_{6,7}$ and $L_{8,9}$ for Krüger-60 in the absence of P-R.

q_1	q_2	ϵ'	ϵ	$L_{6,7}$	$L_{8,9}$
0.51	- 0.90	0.01	0.02	$\pm 0.16267i, \pm 0.77447i, \pm 1.23099i$ (x_0, z_0) = (0.0937801, ± 1.42673)	$\pm 0.032966i, \pm 0.84311i, \pm 1.1966i$ (x_0, z_0) = (0.0157331, ± 2.69349)
0.509	- 0.899	0.03	0.04	$\pm 0.16492i, \pm 0.74891i, \pm 1.2955i$ (x_0, z_0) = (0.093566, ± 1.41668)	$\pm 0.033700i, \pm 0.81075i, \pm 1.26895i$ (x_0, z_0) = (0.015876, ± 2.66558)
0.508	- 0.898	0.05	0.06	$\pm 0.16715i, \pm 0.72843i, \pm 1.35494i$ (x_0, z_0) = (0.0933518, ± 1.40688)	$\pm 0.034438i, \pm 0.78682i, \pm 1.33285i$ (x_0, z_0) = (0.016020, ± 2.63834)
0.507	- 0.897	0.07	0.08	$\pm 0.16936i, \pm 0.71128i, \pm 1.41103i$ (x_0, z_0) = (0.0931375, ± 1.39734)	$\pm 0.035180i, \pm 0.76776i, \pm 1.3919i$ (x_0, z_0) = (0.016162, ± 2.61173)

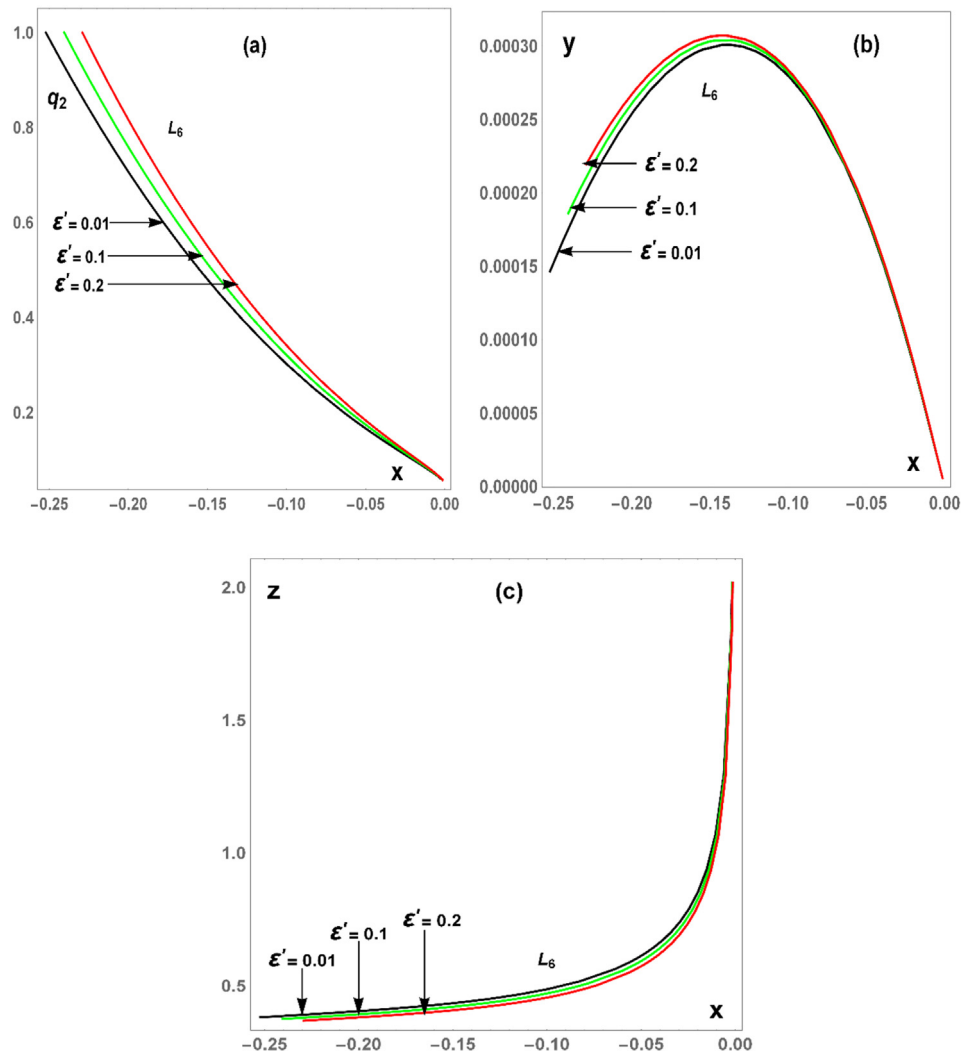


Figure 7. Variation of the equilibrium solutions for L_6 (the situation is same at the symmetric point L_7) of RW-Monocerotis ($\mu = 0.2857$) system for fixed values of $q_1 = -0.02$, $c_d = 1125.18$ and for various values of the centrifugal parameter $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line) and $\epsilon' = 0.2$ (red line), respectively for (a) x -coordinate vs. q_2 , (b) x - vs. y - coordinate, and (c) x - vs. z - coordinate as functions of the parameter q_2 in the interval $(0, 1]$.

More so, in the absence of P-R drag, i.e. $W_1 = 0 = W_2$, then the partial derivatives of (21) yield Eq. (25) as follows:

$$A_1 = B_1 = C_1 = 0, A_3 = B_3 = C_3 = 0, A_5 = B_5 = C_5 = 0, A_4 = B_2 = B_6 = C_4 = 0,$$

$$A_2 = \Omega_{xx}^{(0)} = \beta - \frac{Q_1}{r_{01}^3} - \frac{Q_2}{r_{02}^3} + 3 \left[\frac{Q_1(x_0 + \mu)^2}{r_{01}^5} + \frac{Q_2(x_0 + \mu - 1)^2}{r_{02}^5} \right],$$

$$C_2 = A_6 = \Omega_{xz}^{(0)} = 3 \left[\frac{Q_1(x_0 + \mu)}{r_{01}^5} + \frac{Q_2(x_0 + \mu - 1)}{r_{02}^5} \right] z_0,$$

$$B_4 = \Omega_{yy}^{(0)} = \beta - \frac{Q_1}{r_{01}^3} - \frac{Q_2}{r_{02}^3} + 3 \left[\frac{Q_1}{r_{01}^5} + \frac{Q_2}{r_{02}^5} \right] y_0^2, \quad (25)$$

$$C_6 = \Omega_{zz}^{(0)} = -\frac{Q_1}{r_{01}^3} - \frac{Q_2}{r_{02}^3} + 3 \left[\frac{Q_1}{r_{01}^5} + \frac{Q_2}{r_{02}^5} \right] z_0^2$$

In this case the characteristic equation is thus

$$\lambda^6 + f_4 \lambda^4 + f_2 \lambda^2 + f_0 = 0, \quad (26)$$

where

$$f_4 = 4\alpha^2 - A_2 - B_4 - C_6, \quad f_2 = A_2 B_4 + B_4 C_6 + A_2 C_6 - 4\alpha^2 C_6 - A_6 C_2, \\ f_0 = A_6 B_4 C_2 - A_2 B_4 C_6 \quad (\text{see also Kalantonis et al., 2021}) \text{ and the obtained eigenvalues determine the stability or instability of the respective equilibrium point.}$$

5. Numerical application

Next, we compute numerically, the effects of various parameters involved on the positions and stability of the out of plane equilibria L_i , $i = 6, 7, 8, 9$ for the binary systems RW-Monocerotis and Krüger-60. The astrophysical data of these binary systems are borrowed from Das et al. (2008). The binary system RW-Monocerotis has the masses of the bigger primary and smaller primary $M_1 = 2.5M_\odot$ and $M_2 = 1.0M_\odot$ and luminosities $L_1 = 52L_\odot$ and $L_2 = 8.3L_\odot$, respectively. It has dimensionless velocity of light $C_d = 1125.18$. Those of the Krüger-60 are $M_1 = 0.27M_\odot$, $M_2 = 0.16M_\odot$, $L_1 = 0.010L_\odot$, $L_2 = 0.0034L_\odot$, $C_d = 47240.17$. It is important to note that out-of-plane equilibria do not exist for any combination of the problem parameters. For this reason, and from now on, we will consider sets of $(q_1, q_2, \alpha, \beta, \mu)$ such that conditions (8) and (13) and

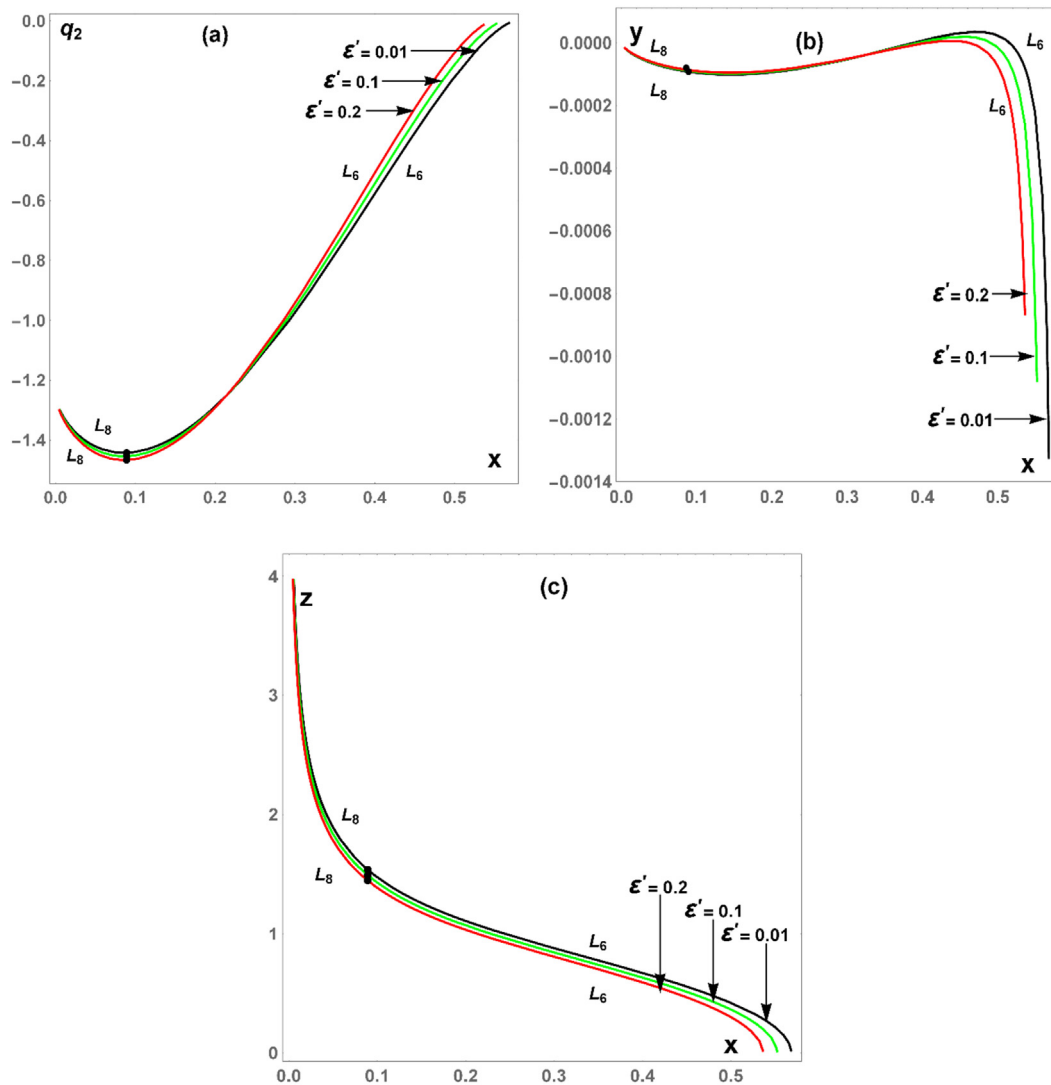


Figure 8. Variation of the equilibrium solutions for L_6 and L_8 (the situation is same at the symmetric point L_7 and L_9 , respectively) of RW-Monocerotis ($\mu = 0.2857$) system for fixed values of $q_1 = 0.5$, $c_d = 1125.18$ and for various values of the centrifugal parameter $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line) and $\epsilon' = 0.2$ (red line), respectively for (a) x - coordinate vs. q_2 , (b) x - vs. y - coordinate, and (c) x - vs. z - coordinate as functions of the parameter q_2 in the interval $[-1.5, 0)$. Note that the symbol $\cdot$ represents where L_6 coincides with L_8 .

conditions (8) and (18) are always satisfy in the cases of absence and presence of P-R drag effect, respectively.

5.1. Positions and stability of out of plane equilibria in the absence of P-R drag effect

In this subsection we will study the existence, location and stability of the out of plane equilibrium points of the problem under the effects of radiation pressure and small perturbations in the Coriolis and centrifugal forces for the binary systems RW-Monocerotis ($\mu = 0.2857$) and Krüger-60 ($\mu = 0.3721$). To investigate the influence of the radiation pressure and small perturbation in the centrifugal force on the positions of the equilibria for the binary system RW-Monocerotis, the radiation pressure of the first primary is set to be $q_1 = -0.02$ while that of the second primary q_2 varies for various values of the small perturbation in the centrifugal force ϵ' . In Figure 5, we present the variation of the equilibrium solutions for L_6 (the situation is same at the symmetric point L_7) in (x, q_2) and (x, z) , correspondingly, for $q_1 = -0.02$ and for different values of $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line), and $\epsilon' = 0.2$ (red line) while q_2 increases from 1 to zero. As can be seen, with the increase of the

radiation pressure and small perturbation in the centrifugal parameter (in Figure 5a, b), the positions of the symmetrical equilibrium points ($x_0, \pm z_0$) move away from the primaries and approach the Oz - axis. The situation is analogous for other fixed negative values of q_1 . Recall here that radiation pressure increases when $q_{1,2}$ decreases. Similarly, when the radiation pressure of the first primary is set to be $q_1 = 0.5$ while that of the second primary q_2 varies for the same numerical values of ϵ' , we present in Figure 3a, b where we plot the variation of the two positive equilibrium points, L_6, L_8 in (x, q_2) and (x, z) , correspondingly as a function of the parameter q_2 , which varies in the interval $[-1.5, 0)$ for fixed values of $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line), and $\epsilon' = 0.2$ (red line). It is observed that when the radiation pressure of the first primary does not overwhelm the gravitational one there are at most two pairs of such equilibria. So, by decreasing the values of the radiation factor when the centrifugal parameter ϵ' increases, the positions of out of plane equilibria move away from the primary and approach the Oz - axis.

Figures 4 and 6 illustrate the variation of the equilibrium solutions for Krüger-60 as q_2 varies for the same numerical values of ϵ' and q_1 as used in generating Figures 3 and 5. In Figure 4a, b we plot the position of L_6 in (x, q_2) and (x, z) , correspondingly as functions of a small perturbation in the

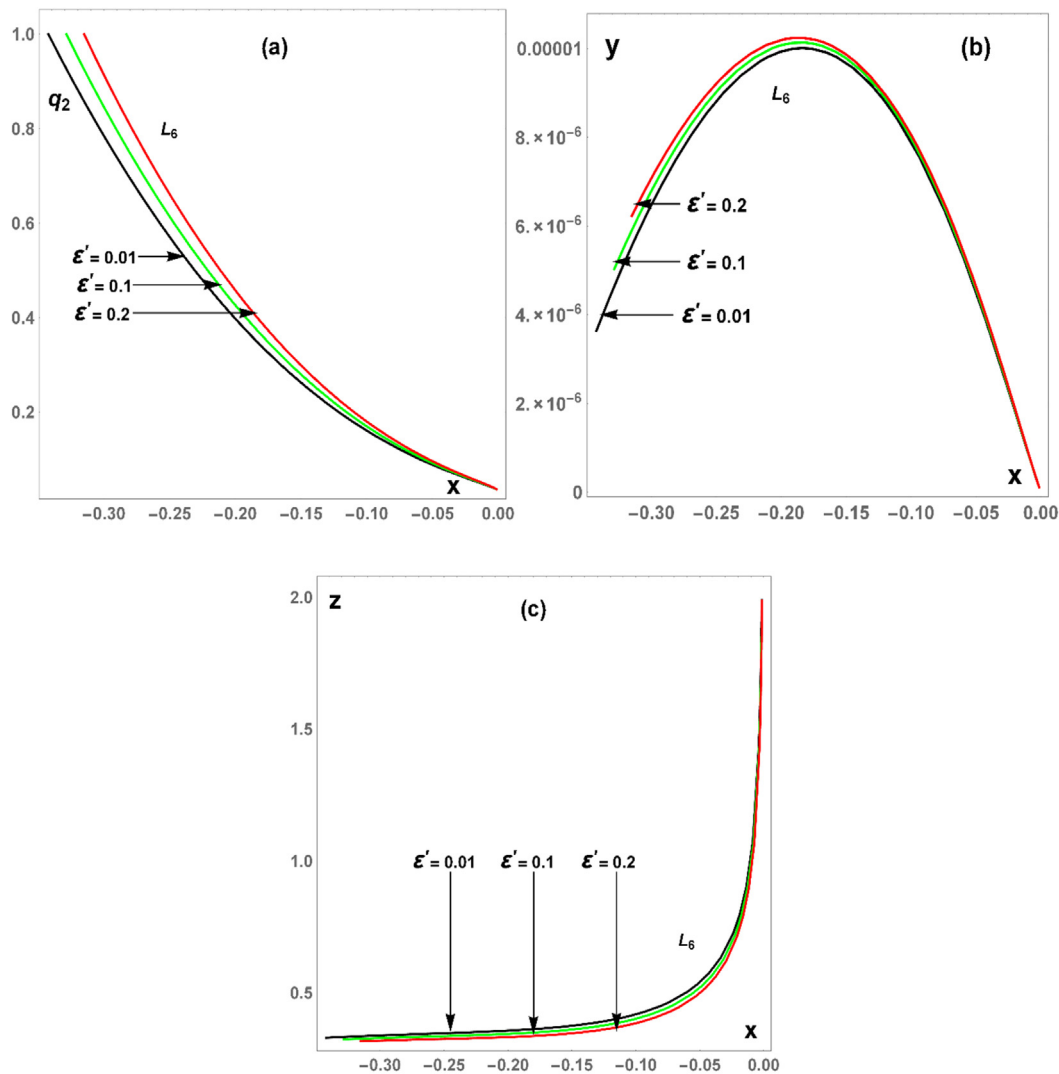


Figure 9. Variation of the equilibrium solutions for L_6 (the situation is same at the symmetric point L_7) of Krüger-60 ($\mu = 0.3721$) system for fixed values of $q_1 = -0.02$, $c_d = 47240.17$ and for various values of the centrifugal parameter $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line) and $\epsilon' = 0.2$ (red line), respectively for (a) x -coordinate vs. q_2 , (b) x - vs. y - coordinate, and (c) x - vs. z - coordinate as functions of the parameter q_2 in the interval $(0, 1]$.

centrifugal force ($\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line), and $\epsilon' = 0.2$ (red line)) and radiation factors ($q_1 = -0.02$, q_2 in the interval $[1, 0)$) while Figure 6a and Figure 6b present the positions of two positive out-of-plane equilibria L_6, L_8 for small perturbation in the centrifugal force ($\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line), and $\epsilon' = 0.2$ (red line)) and radiation factors ($q_1 = 0.5$, q_2 in the interval $[-0.91, 0)$) for the binary system Krüger-60. We notice that the effects of q_1 , q_2 , and ϵ' parameters on the positions of the out-of-plane equilibria are the same behavior with its effects on the out-of-plane equilibria of RW-Monocerotis even though the orbital properties are different. However, comparing Figures 3 and 5 with Figures 4 and 6, we observe that for positive q_1 , the interval of q_2 for which the equilibria exist increase (i.e., q_2 decreases) with the mass parameter, μ and small perturbation in the centrifugal force, ϵ' . The opposite is true when q_1 is negative. More so, when q_1 is positive, as μ increases, the range of variation of the x -coordinate of the equilibrium points is more narrow. The opposite is true when q_1 is negative. From these figures, it is obvious, that the mass parameter, radiation factors and small perturbation in the centrifugal force have significant effects on the motion of an infinitesimal body near the out-of-plane equilibria as determined by the existence and positions of these equilibria.

Next, in Tables 1 and 2, we compute the positions of the out-of-plane equilibria together with the roots of the characteristic equation (Equation

26) for different values of radiation factors and small perturbations in the Coriolis and centrifugal forces for the binary systems RW-Monocerotis and Krüger-60, respectively. An analysis of the numerical results reveals the existence of purely imaginary roots at equilibria $L_{6,7}$ leading thus to stability while for the equilibria $L_{8,9}$ we get two opposite real roots and four imaginary which means that due to the real roots these equilibria are unstable.

5.2. Positions and stability of out of plane equilibria in the presence of P-R drag effect

In this subsection, we shall discuss the existence, locations and stability of the out-of-plane equilibria $L_i, i = 6, 7, 8, 9$ for various values of small change in the Coriolis parameter ϵ and centrifugal perturbation ϵ' and radiation factors q_1, q_2 with effective P-R drag force. In Figures 7 and 8, the changes in the variation of the out of plane equilibria for RW-Monocerotis are investigated as regards the effects of the radiation factors q_1, q_2 and small perturbation in the centrifugal force ϵ' . In particular, in Figure 7a, b and c we present the variation of the positive equilibrium solutions L_6 in (x, q_2) , (x, y) , and (x, z) , correspondingly, for fixed value of $q_1 = -0.02$ when the small perturbation in the centrifugal force takes the values $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line), and $\epsilon' = 0.2$ (red line)

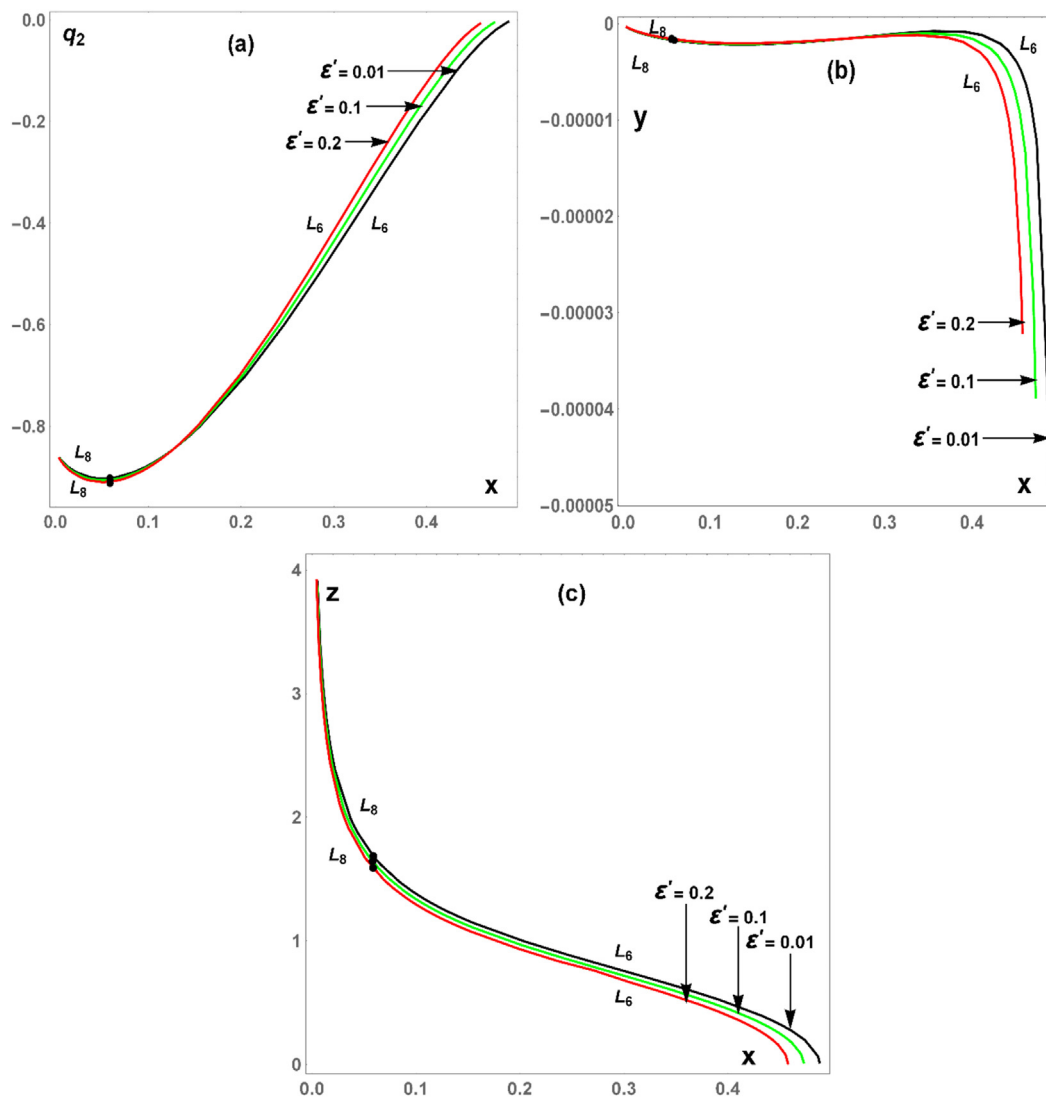


Figure 10. Variation of the equilibrium solutions for L_6 and L_8 (the situation is same at the symmetric point L_7 and L_9 , respectively) of Krüger-60 ($\mu = 0.3721$) system for fixed values of $q_1 = 0.5$, $c_d = 47240.17$ and for various values of the centrifugal parameter $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line) and $\epsilon' = 0.2$ (red line), respectively for (a) x - coordinate vs. q_2 , (b) x - vs. y - coordinate, and (c) x - vs. z - coordinate as functions of the parameter q_2 in the interval $[-0.91, 0]$. Note that the symbol $\cdot$ represents where L_6 coincides with L_8 .

Table 3. Locations of out-of-plane equilibria $L_{6,7}$ without P-R and with P-R as a function of ϵ' for $q_1 = -0.03$ and $q_2 = 0.25$ of the binary system RW-Monocerotis, $\mu = 0.2857$, $c_d = 1125.18$

ϵ'	$L_{6,7}$ (Without P-R)	$L_{6,7}$ (With P-R)
	$(x_0, y_0, \pm z_0)$	$(x_0, y_0, \pm z_0)$
0	$(-0.0687448880, 0, \pm 0.6423905207)$	$(-0.0687450803, 0.0001632067, \pm 0.6423908080)$
0.01	$(-0.0682449689, 0, \pm 0.6415889795)$	$(-0.0682451594, 0.0001626739, \pm 0.6415892645)$
0.02	$(-0.0677528389, 0, \pm 0.6407985666)$	$(-0.0677530274, 0.0001621387, \pm 0.6407988491)$
0.03	$(-0.0672683076, 0, \pm 0.6400190346)$	$(-0.0672684943, 0.0001616012, \pm 0.6400193149)$
0.04	$(-0.0667911912, 0, \pm 0.6392501443)$	$(-0.0667913760, 0.0001610618, \pm 0.6392504221)$
0.05	$(-0.0663213117, 0, \pm 0.6384916631)$	$(-0.0663214948, 0.0001605208, \pm 0.6384919387)$

Table 4. Locations of out-of-plane equilibria $L_{6,7}$ without P-R and with P-R as a function of ϵ' for $q_1 = -0.03$ and $q_2 = 0.25$ of the binary system Krüger-60, $\mu = 0.3721$, $c_d = 47240.17$

ϵ'	$L_{6,7}$ (Without P-R)	$L_{6,7}$ (With P-R)
	$(x_0, y_0, \pm z_0)$	$(x_0, y_0, \pm z_0)$
0	$(-0.1317931059, 0, \pm 0.4643485339)$	$(-0.1317931061, 0.0000063735, \pm 0.4643485342)$
0.01	$(-0.1310501803, 0, \pm 0.4631197526)$	$(-0.1310501805, 0.0000063709, \pm 0.4631197529)$
0.02	$(-0.1303178251, 0, \pm 0.4619040854)$	$(-0.1303178253, 0.0000063679, \pm 0.4619040857)$
0.03	$(-0.1295957910, 0, \pm 0.4607012707)$	$(-0.1295957912, 0.0000063645, \pm 0.4607012710)$
0.04	$(-0.1288838372, 0, \pm 0.4595110547)$	$(-0.1288838374, 0.0000063606, \pm 0.4595110550)$
0.05	$(-0.1281817305, 0, \pm 0.4583331907)$	$(-0.1281817307, 0.0000063563, \pm 0.4583331910)$

Table 5. The eigenvalues $\lambda_{1,2}$, $\lambda_{3,4}$, $\lambda_{5,6}$ of Eq. (23) and the corresponding positions $(x_0, y_0, \pm z_0)$ of the out-of-plane equilibria $L_{6,7}$ and $L_{8,9}$ for RW-Monocerotis, $\mu = 0.2857$, $c_d = 1125.18$

q_1	q_2	ϵ'	ϵ	$L_{6,7}$	$L_{8,9}$
− 0.02	1	0.01	0.02	0.523586 ± 0.965709i, − 0.00507067 ± 0.910379i, − 0.527354 ± 0.965622i (− 0.252201, 1.4658 × 10 ^{−4} , ± 0.381311)	–
	0.5	0.1	0.11	0.159893 ± 0.807838i, − 0.00719459 ± 1.22059i, − 0.159501 ± 0.798114i (− 0.146948, 3.3037 × 10 ^{−4} , ± 0.423274)	–
	0.2	0.2	0.23	0.0006260 ± 0.682554i, − 0.0023154 ± 1.74579i, − 0.0024032 ± 0.37139i (− 0.056086, 2.0275 × 10 ^{−4} , ± 0.550558)	–
0.5	− 0.5	0.1	0.02	0.687756 ± 0.705273i, − 0.0012826 ± 1.44711i, − 0.688754 ± 0.705307i (0.418566, 1.7789 × 10 ^{−5} , ± 0.630616)	–
	− 1.3	0.1	0.11	0.277747 ± 0.498096i, − 0.00066733 ± 1.54514i, − 0.278347 ± 0.497468i (0.19686, − 9.226 × 10 ^{−5} , ± 1.08085)	0.020450, − 0.0205612, − 0.0000171528 ± 0.746223i, − 0.000039439 ± 1.47376i (0.0051617, − 1.7883 × 10 ^{−5} , ± 3.96646)
	− 1.4	0.2	0.23	0.149149 ± 0.427987i, − 0.00056369 ± 1.82505i, − 0.149719 ± 0.426936i (0.153261, − 9.51352 × 10 ^{−5} , ± 1.16784)	0.0756615, − 0.0760295, − 0.000036110 ± 0.665303i, − 0.000153321 ± 1.79297i (0.028967, − 5.06735 × 10 ^{−5} , ± 2.15107)

as q_2 increases from 1 to zero while Figure 8a, b and c show the variation of the two positive out-of-plane equilibria L_6, L_8 in (x, q_2) , (x, y) , and (x, z) , correspondingly, as functions of q_2 in the interval $[-1.5, 0)$ for fixed

values of $\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line), and $\epsilon' = 0.2$ (red line) but for $q_1 = 0.5$. Evidently, the variational trends of the corresponding positions are similar to the scenario without P-R drag as

Table 6. The eigenvalues $\lambda_{1,2}$, $\lambda_{3,4}$, $\lambda_{5,6}$ of Eq. (23) and the corresponding positions $(x_0, y_0, \pm z_0)$ of the out-of-plane equilibria $L_{6,7}$ and $L_{8,9}$ for Krüger-60 $\mu = 0.3721$, $c_d = 47240.17$

q_1	q_2	ϵ'	ϵ	$L_{6,7}$	$L_{8,9}$
− 0.02	1	0.01	0.02	0.638182 ± 0.959439i, − 0.0001331 ± 1.05606i, − 0.638296 ± 0.959435i (− 0.342342, 3.6422 × 10 ^{−6} , ± 0.329827)	–
	0.5	0.1	0.11	0.377156 ± 0.863629i, − 0.0002202 ± 1.23341i, − 0.37714 ± 0.863562i (− 0.22104, 9.79109 × 10 ^{−6} , ± 0.340393)	–
	0.2	0.2	0.23	0.152971 ± 0.60422i, − 0.000115765 ± 1.72291i, − 0.153004 ± 0.604078i (− 0.10964, 8.53491 × 10 ^{−6} , ± 0.374124)	–
0.5	− 0.5	0.1	0.02	0.496849 ± 0.655104i, − 0.000026569 ± 1.33305i, − 0.496867 ± 0.655097i (0.284201, − 1.30028 × 10 ^{−6} , ± 0.79425)	–
	− 1.3	0.1	0.11	3.9397 × 10 ^{−6} ± 0.640483i, − 0.000011650 ± 1.49691i, − 0.000015488 ± 0.278288i (0.116234, − 2.07586 × 10 ^{−6} , ± 1.25762)	0.0115139, − 0.0115161, − 3.45438 × 10 ^{−7} ± 0.746442i − 7.48886 × 10 ^{−7} ± 1.47355i (0.003331, − 3.05889 × 10 ^{−7} , ± 4.39254)
	− 1.4	0.2	0.23	1.96809 × 10 ^{−8} ± 0.427987i, − 8.67424 × 10 ^{−6} ± 1.79853i, − 0.00001024 ± 0.131665i (0.079063, − 1.82736 × 10 ^{−6} , ± 1.42023)	0.0487527, − 0.0487616, − 9.12755 × 10 ^{−7} ± 0.667121i − 3.69962 × 10 ^{−6} ± 1.79135i (0.026379, − 1.07861 × 10 ^{−6} , ± 2.11123)

described previously in Figures 3 and 5. However, another difference, w.r.t. the previous cases, is the small but finite y components of the various equilibria of the system which is as a result of P-R drag effect.

Similarly, Figures 9 and 10 illustrate the variation of out-of-plane equilibria for Krüger-60 at different values of q_2 but for the same numerical values of ϵ' and fixed value of q_1 as used in generating Figures 7 and 8. In particular, Figure 9a, b and c introduces a small perturbation in the centrifugal force ($\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line), and $\epsilon' = 0.2$ (red line)) and radiation factors ($q_1 = -0.02$, q_2 in the interval $[1, 0)$) on the equilibrium solutions of L_6 in (x, q_2) , (x, y) , and (x, z) , correspondingly while Figure 10a, b and c present the variation of two positive out-of-plane equilibria L_6, L_8 in (x, q_2) , (x, y) , and (x, z) , correspondingly, for a small perturbation in the centrifugal force ($\epsilon' = 0.01$ (black line), $\epsilon' = 0.1$ (green line), and $\epsilon' = 0.2$ (red line)) and radiation factors ($q_1 = 0.5$, q_2 in the interval $[-0.91, 0)$) for the binary system Krüger-60. We notice that the effects of q_1 , q_2 , and ϵ' parameters on the positions of the out-of-plane equilibria are the same behavior with its effects on the out-of-plane equilibria of RW-Monocerotis even though the orbital properties are different. However, comparing Figures 7 and 8 with Figures 9 and 10, we observe that when q_1 is positive, as μ increases, the range of variation of the x -coordinate of the equilibrium points is more narrow while at the same time they-coordinate is wider. The opposite is true when q_1 is negative.

Finally, putting together these results, it is also observed from Figures 7, 8, 9, 10 that the variational trend of the corresponding positions of these equilibria vary in a relatively small range with P-R drag compared to that without P-R drag (Figures 3, 4, 5, 6) as the radiation factors and small perturbation in the centrifugal force vary. From the results we can conclude that the P-R perturbing forces under consideration have no significant effect on the locations of the various equilibrium points $L_i, i = 6, 7, 8, 9$ in the vicinity of the binary systems under consideration. Despite of P-R drag forces are very or tiny small effect, there is no guarantee that the dynamical properties are unaffected. These effects can easily be seen in tables. As a particular example we compute the coordinates of the equilibria $L_{6(7)}$ which are shown in Tables 3 and 4 for the binary systems RW-Monocerotis and Krüger-60, respectively, as a function of ϵ' for fixed values $q_1 = -0.03$ and $q_2 = 0.25$. The analysis reveals that the corresponding equilibria values are close each other. This indicates the P-R drag terms have negligible effect.

Next, the stability of out-of-plane equilibria are obtained by substituting the evaluated values at equilibrium (these quantities are given in Eq. (24)) into the characteristic Eq. (23). The characteristic roots as well as the corresponding positions of the equilibria obtained are shown in Tables 5 and 6 for the binary systems RW-Monocerotis and Krüger-60, respectively for a wide range of the various parameters q_1, q_2, ϵ and ϵ' . We observe that the equilibria $L_{6,7}$ exist for certain values of radiation parameters q_1 and q_2 for the binary systems under consideration. More so, certain radiation parameters exist for which it is possible to have four equilibria, $L_i, i = 6, 7, 8, 9$ for these binary system. Analysis of the numerical results reveals no case in which the roots are all imaginary. This provides numerical evidence that the out of plane equilibrium points, in all cases, are generally unstable. This agrees with the studies of Ragos et al. (1995) and Das et al. (2009).

6. Discussion and conclusions

We have studied the motion of an infinitesimal body under the PR-effects of both luminous primaries together with small perturbations in the Coriolis and centrifugal forces and found that the equations of motion given in the literature allow the existence of out of the orbital plane equilibrium points. It was also found that the positions of these equilibrium points are independent of the effect of small perturbation in the Coriolis force. We split the study in two cases, first when only the major radial component of the pressure force is considered and second when the aberrational deceleration due to P-R effect is incorporated. We

found that two $L_{6,7}$ or four $L_{6,7,8,9}$ such points (depends on $q_{1,2}$) may lie in symmetrical positions with respect to the orbital plane of the primaries. Figures 3, 4, 5, 6 and Figures 7, 8, 9, 10 give the variation of the equilibrium solutions without and with P-R drag, respectively of the RW-Monocerotis and Kruger-60 binary systems. It was found that four equilibria $L_i, i = 6, 7, 8, 9$ can exist when the radiation pressure of the smaller primary q_2 exceeds its gravitational one q_1 . In the opposite case we have two equilibria $L_{6,7}$. It was observed from Figures 3, 4, 5, 6 and Figures 7, 8, 9, 10 that the positions of these equilibria shift slightly off the orbital plane with an increase in system parameters. However, the P-R drag has no significant changes on the locations of out of plane equilibria for the binary systems under consideration when comparing Figures 3, 4, 5, 6 (without P-R) and Figures 7, 8, 9, 10 (with P-R drag). This is confirmed from Tables 3 and 4 with and without relativistic terms that all the corresponding values are close each other. Despite of P-R drag forces are very or tiny small effect, there is no guarantee that the dynamical properties are unaffected. Additionally, it was observed that the involved parameters of the problem not only affect the positions of the corresponding equilibria but they influence their stability as well. For all the considered cases, it was found that all equilibria are unstable due to the existence of a positive real root or a complex root with positive real part (Tables 5 and 6) except for the equilibria L_6 and L_7 in the case of absence of P-R drag (Tables 1 and 2) where we got purely imaginary roots parts which means that these points are stable. Therefore, we conclude that stability of the equilibrium points L_6 and L_7 is affected from the P-R drag force of the primaries as they are found to be unstable when considering P-R drag effect but conditionally stable in the absence of P-R drag as seen in aforementioned Tables. This agrees with the studies of Ragos et al. (1995) and Das et al. (2009) and references therein.

Declarations

Author contribution statement

Aguda Ekele Vincent: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Jagadish Singh: Conceived and designed the experiments; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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